R. McCallum.

SOLUTIONS OF THE EXERCISES

IN

MODERN GEOMETRY

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SOLUTIONS OF THE EXERCISES

IN

MODERN GEOMETRY

BY

C. GODFREY AND A. W. SIDDONS

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KEY TO THE EXERCISES

CHAPTER I.

2. Let $\overline{AB}=b$, $\overline{AC}=c$, $\overline{AD}=d$. Then $\overline{CD}=d-c$, $\overline{DB}=b-d$, $\overline{BC}=c-b$.

$$\therefore \overline{AB}\,.\,\overline{CD}+\overline{AC}\,.\,\overline{DB}+\overline{AD}\,.\,\overline{BC}=b(d-c)+c(b-d)+d(c-b)$$
$$\equiv 0.$$

3. $m\,.\,\overline{OA}+n\,.\,\overline{OB}=m(\overline{OC}-\overline{AC})+n(\overline{OC}+\overline{CB})$
$$=(m+n)\,\overline{OC}.$$

4. Let $\overline{OA}=a$, etc.; then $\overline{BC}=c-b$, etc.; $\overline{OP}=\dfrac{b+c}{2}$, etc.

$$\therefore \Sigma\,\overline{OP}\,.\,\overline{BC}=\Sigma\frac{b+c}{2}(c-b)\equiv 0.$$

5. From Ex. 2, $\dfrac{\overline{AB}\,.\,\overline{CD}}{\overline{AD}\,.\,\overline{BC}}+\dfrac{\overline{AC}\,.\,\overline{DB}}{\overline{AD}\,.\,\overline{BC}}+1=0$, $\therefore$ etc.

6. Let $PA=a$, $PB=b$, $PX=x$, $PY=y$.

Then $PA\,.\,BX\,.\,BY=a(x-b)(y-b)$
$$=axy-aby-abx+ab^2.$$

And $PB\,.\,AX\,.\,AY=b(x-a)(y-a)=bxy-aby-abx+a^2b$
$$=ab^2-aby-abx+axy.$$

CHAPTER II.

7. C becomes the point at infinity on the bisector of $\angle$ FAE (*not* the point at infinity on AE). So for B. EC, DB become the line at infinity.

8. The circumference becomes a finite line *plus* the line at infinity. Let PQ be the chord, and O a point on the circumference.

   (i) If P, Q are finite points on the finite line, and O on the finite line, then $\angle$ POQ is 0° or 180°.

   (ii) If P, Q are as above, and O on the line at infinity, PO, QO are $\parallel$, $\therefore \angle$ POQ = 0.

   (iii) If P, Q are points on the line at infinity, and O on the finite line, PO, QO are fixed in direction, though PO cannot be distinguished from OP, or QO from OQ. Hence PQ still subtends equal or supplementary angles at O.

   (iv) If P, Q are as in (iii), and O on the line at infinity, $\angle$ POQ becomes indefinite.

CHAPTER III.

9. The centroid of A, B, C, D is the mid-point of the join of the mid-points of AB, CD. Similarly of AC, BD; and of AD, BC. These three points are therefore identical.

10. The centroid of A, B, C, D is on Aα and divides it in the ratio 3 : 1. So for Bβ, Cγ, Dδ.

11. (i) The 3 joins of mid-points of opposite edges of a tetrahedron are concurrent and are bisected at the point of concurrence.

    (ii) The 4 lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent, and are divided in the ratio 3 : 1.

12. A particular case of Th. 3. Take G as origin, and the line through G as axis of x.

13. Let M be the projection of P on OG. Then, with a suitable convention for the sign of GM, $OP^2 = PG^2 + OG^2 + 2OG \cdot GM$.
$\therefore \Sigma OP^2 = \Sigma PG^2 + n \cdot OG^2 + 2OG\,\Sigma GM$.
But $\Sigma GM = 0$, by Ex. 12.

CHAPTER IV.

21.
$$\frac{PB}{PA}\cdot\frac{QC}{QB}\cdot\frac{RD}{RC}\cdot\frac{SA}{SD}$$
$$= \frac{\sin PAB}{\sin PBA}\cdot\frac{\sin QBC}{\sin QCB}\cdot\frac{\sin RCD}{\sin RDC}\cdot\frac{\sin SDA}{\sin SAD}$$
$= 1$ ($\because \angle PAB = \angle SAD$, etc.).

22. Let bisector of $\angle A$ meet BC in D. Then
$BD : AB = \sin BAD : \sin BDA$; and $CD : AC = \sin CAD : \sin CDA$
But $\sin BDA = \sin CDA$, $\therefore$ $BD : AB = CD : AC$.

25. $AD^2 + AB^2 = 2AC^2 + 2BC^2$, $\therefore$ etc.

26. $QS^2 + QP^2 = 2QR^2 + 2PR^2$, $\therefore$ etc.

27. $OA^2 + OC^2 = 2OB^2 + 2BC^2$, $OD^2 + OB^2 = 2OC^2 + 2BC^2$.
$\therefore$ $OA^2 + OC^2 + 2OD^2 + 2OB^2 = 2OB^2 + 2BC^2 + 4OC^2 + 4BC^2$.
$\therefore$ $OA^2 + 2OD^2 = 3OC^2 + 6BC^2$.

28. $OA^2 + OC^2 = 2OB^2 + 2BC^2$, $OB^2 + OD^2 = 2OC^2 + 2BC^2$.
$\therefore$ $OA^2 + OC^2 + OB^2 + OD^2 = 2OB^2 + 4BC^2 + 2OC^2$.
$\therefore$ $OA^2 + OD^2 = OC^2 + OB^2 + 4BC^2$.

29. Let N be the projection of A on BC: let $\angle AQB$ be acute.
$AB^2 = AQ^2 + BQ^2 - 2BQ \cdot NQ$ (neglecting sign).
$AC^2 = AQ^2 + CQ^2 + 2QC \cdot NQ$.
$\therefore$ $AB^2 + n \cdot AC^2 = (n+1)\,AQ^2 + BQ^2 + n \cdot CQ^2$.

30. By Apollonius, the distance of the point from the mid-point of AB is constant.

32. Let P, Q be the mid-points of AC, BD.

$AB^2 + AD^2 = 2BQ^2 + 2AQ^2$, $CB^2 + CD^2 = 2BQ^2 + 2CQ^2$.

$\therefore\ AB^2 + AD^2 + CB^2 + CD^2 = BD^2 + 2AQ^2 + 2CQ^2$

$= BD^2 + AC^2 + 4PQ^2$.

33. The mid-points of sides are the vertices of a parallelogram, apply Ex. 31 to this.

34. Let the sides be a, b, c; the medians x, y, z. Then

$$b^2 + c^2 = 2x^2 + \frac{a^2}{2},\quad c^2 + a^2 = 2y^2 + \frac{b^2}{2},\quad a^2 + b^2 = 2z^2 + \frac{c^2}{2}.$$

Now add; $2\Sigma a^2 = 2\Sigma x^2 + \frac{1}{2}\Sigma a^2$, $\therefore$ etc.

$\therefore\ 3\Sigma a^2 = 4\Sigma x^2$.

35. A, B, C, are the mid-points of B′C′, C′A′, A′B′; and the altitudes of $\triangle$ ABC are the perp. bisectors of the sides of $\triangle$ A′B′C′.

36. Let hexagon be AB′CA′BC′. SBA′C is a rhombus; and the sides BA′, A′C are each equal to R. So are the other sides of the hexagon. Also $\angle$ C′AB′ = $\angle$ BSC = $\angle$ BA′C.

37. Theorem 11.

38. "If a polygon is such that a circle can be described about it, the perpendicular bisectors of the sides are concurrent."

39. Let the diameter through the intersection of the tangents cut the circle in P, Q; P being nearer to intersection. Inscribe (or escribe) a circle to $\triangle$ made by the two tangents, and the tangent at P (or Q).

(The construction may also be effected by describing a ⊙ to touch the given ⊙ and one of the tangents at the point of contact of the latter.)

40. The chords are equidistant from the centre, and therefore equal.

41. It is necessary to prove that ZP bisects $\angle$ AZY.

Now $\angle$ AZP = $\angle$ PYZ in alternate segment $_$ PZY, by symmetry about AI, $\therefore$ etc.

42. Since AK bisects $\angle$ BAC internally, arc BK = arc CK. Similarly arc BK′ = arc CK′. $\therefore$ KK′ bisects BC at rt. $\angle$s.

43. U is mid-point of arc BC, $\therefore$ etc.

44. AII_1 is internal bisector of $\angle BAC$.

45. I_2AI_3 is external bisector of $\angle BAC$.

46. Internal and external bisectors are at rt. $\angle$ s.

47. The interior common tangents to two $\odot$ s intersect on the line of centres.

48. Use Theorems 5, 11, 12.

49. YZ is equally inclined to AB, AC; and $\therefore$ is $\perp$ to I_1I_2, $\therefore$ etc.

50. (i) $IBI_1 = 90°$.

(ii) Describe $\odot$ to pass through B, C, with $\frac{II_1}{2}$ as radius.

Draw a line from B making $\angle \frac{B}{2}$ with BC; where this cuts $\odot$ is I. Now draw in-circle; tangents from B, C give A.

51–58. See Theorem 13.

59. 6, 4 ft.

61. Let $\angle A$ be the rt. $\angle$.

Then $r = AY = s - a = \frac{b + c - a}{2}$.

63. Taking AB, AC as the directions of the fixed sides and BC as the moving side, AY_1 and AZ_1 are constant, $\therefore$ the escribed $\odot I_1$ is fixed.

64, 65. See Theorem 14.

68. $\triangle GBa = \triangle GCa$, $\triangle ABa = \triangle ACa$. $\therefore \triangle GAB = \triangle GAC$.

69. Let AP meet QR in X. Then AX : XD $=$ AQ : QB. $\therefore$ AX $= \frac{1}{2}$XD.
Again, $\triangle$s XPR, DPB are similar.
$\therefore$ XP : PD $=$ XR : DB $= 1 : 3$, $\therefore$ PD $= 3$XP.
$\therefore$ PD : XP : AX $= 3 : 1 : 2$, and AP $=$ PD. $\therefore$ AP : AD $= 1 : 2$.

72. BFA is $\perp$ to HC, etc.

74. A, E, D, B are concyclic. $\therefore$ AH . HD $=$ BH . HE, $\therefore$ etc.

75. $\angle SAC = 90° - \angle AS\beta = 90° - \angle B = \angle BAD$, $\therefore$ etc.

76. AFHE is cyclic, $\therefore \angle BHC = \angle FHE =$ suppl. $\angle A$.

77. Let $\angle$s B, D of quadrilateral ABCD be rt. $\angle$s; AB, DC meet in E; BC, AD in F. Then C is orthocentre of $\triangle AEF$. $\therefore$ AC is $\perp$ to EF.

79. It is necessary to prove that DH bisects $\angle EDF$.
Now $\angle EDH = \angle ECH = 90^\circ - \angle A$; similarly
$\angle FDH = 90^\circ - \angle A$.

80. Let tangent at A be PAQ. Then $\angle PAB = \angle ACB$, in alternate segment; also $\angle AFE = \angle ACB$, $\therefore$ PAQ is $\parallel$ to FE, $\therefore$ etc.

82. See fig. 20. $\triangle BHC \equiv \triangle BXC$, whose circumcircle is $\odot ABC$.

83. If X, Y, Z be the circumcentres of $\triangle$s BHC, CHA, AHB; then $HX = HY = HZ = R$, by Ex. 82.

84. $AD \cdot HD = AD \cdot DX = BD \cdot DC$.

85. (i) *Orthocentre.* $\odot ABC$ is fixed; locus of H is the image, in BC, of locus of X.

(ii) *In-centre.*
$$\angle BIC = 180^\circ - \angle \frac{B}{2} - \angle \frac{C}{2}$$
$$= 180^\circ - \left(90^\circ - \frac{A}{2}\right)$$
$$= 90^\circ + \frac{A}{2}, \text{ which is fixed.}$$

(iii) *Centroid.* Since α is fixed and $\alpha G = \frac{1}{3}\alpha A$, $\therefore$ locus of G is similar and similarly situated to locus of A, α being centre of similitude; see Chapter VIII.

86. HA, $S\alpha$ are $\perp$s from the circumcentres of two similar $\triangle$s upon corresponding sides.

87–90. See Theorem 19.

91. $D\alpha$, $E\beta$, $F\gamma$ are chords of the 9-points $\odot$.

92. If AS, $H\alpha$ meet at V, $\triangle$s AHV, $S\alpha V$ are similar, and $AH = 2S\alpha$, $\therefore$ $AV = 2AS$, $\therefore$ etc.

93. HP, HQ, HR are $=$ and $\parallel$ to $S\alpha$, $S\beta$, $S\gamma$, $\therefore \triangle \alpha\beta\gamma \equiv \triangle PQR$.
Or thus: QR is $\parallel$ to BC and $= \frac{1}{2}$BC; and $\beta\gamma = \frac{1}{2}$BC, etc.

94. SP, Aα are diagonals of parallelogram APαS.

96. Both are congruent with $\triangle A\gamma\beta$.

97. By Ex. 96 $\triangle D\gamma\beta \equiv \triangle \alpha\beta\gamma$, $\therefore$ $\angle \gamma D\beta = \angle \gamma\alpha\beta$, $\therefore$ circumcircle of medial triangle passes through D, and similarly through E, F.

98. Circumcircle of $\triangle$ DEF, the pedal $\triangle$ of HBC, passes through α and the mid-points of HB, HC.

101. Medians of $\triangle$ AHK are Aα, HS.

102, 103. ABC is the pedal $\triangle$ and I the orthocentre of $I_1I_2I_3$.

104. O, S, I are circumcentre, 9-points centre, and orthocentre of $I_1I_2I_3$.

105. Same as 9-points $\odot$ of $\triangle$ ABC.

106. Theorem 20.

107. If a line meets the sides of a $\triangle$ in L, M, N; and the $\perp$s at L, M, N to the sides are concurrent at a point P, then P is on the circumcircle.

Proof. (i) P cannot be inside $\triangle$. For let $\angle$s B, C be acute; and let P be inside $\triangle$. Then LN is inclined from LP towards B; and LM is inclined from LP towards C; and L is between B and C, $\therefore$ LNM cannot be a straight line.

(ii) $\angle$ APN = $\angle$ AMN, $\angle$ BPN = $\angle$ MLC.
$\therefore$ $\angle$ APB = $\angle$ AML + $\angle$ MLC = $180^\circ - \angle$ C.
$\therefore$ P being outside the $\triangle$ is on the circumcircle.

109. (i) The $\perp$ from A to BC. (ii) BC.

110. If $\perp$s from X meet AB, AC in N, M, then
$\angle$ ANM = $\angle$ AXM = $90^\circ - \angle$ XAC = $\angle$ C, $\therefore$ etc.

111. U is mid-point of arc BC, $\therefore$ $\perp$ from U bisects BC.

112. $\angle$ BAU = $\angle$ BPL = $\angle$ BNL, $\therefore$ etc.

113. $\angle$ LPX = $\angle$ PXA = $\angle$ PBA = $\angle$ PLR. Hence, using rt. angled $\triangle$ PLQ, R is mid-point of PQ.

114. HD = XD, $\therefore$ $\angle$ HQD = $\angle$ XQD = $\angle$ RQL = $\angle$ RLQ, $\therefore$ etc.

115. PR = RQ and LRNM is $\parallel$ to QH, $\therefore$ PH is bisected by LRNM.

116. By symmetry PX, pH intersect on BC; now
$$\angle \text{NLP} = \angle \text{NBP} = \angle \text{AXP} = \angle \text{XHQ} = \angle \text{H}p\text{P}.$$
$\therefore$ pH is $\parallel$ to LMN, and PL = Lp, $\therefore$ PH is bisected by LMN.

117. Draw the circumcircles of two of the $\triangle$s formed by the four lines; let P be their other point of intersection. Then P is the point required.

118. See Ex. 117. A parabola may be drawn to touch the four lines; the circumcircle of each of the $\triangle$s formed by 3 of these lines passes through the focus S.

119. Construct a $\odot$ of the given radius, and place in it a chord = the given base. This fixes the vertical $\angle$ as one of two supplementary $\angle$s. Hence the sum of the base $\angle$s. As the difference also is given, the base $\angle$s are determined (in two ways).

120. (i) The locus of the orthocentre is the straight line through C $\perp$ to AB. (ii) The locus of the circumcentre is the $\perp$ bisector of BC. (iii) The locus of the incentre is the two bisectors of $\angle$ B. (iv) The locus of the centroid is a straight line $\parallel$ to AB, cutting BC in a point D so that BD = $\frac{1}{3}$ BC. (v) If the altitude from A meets the circumcircle in X, then X is the image of H in BC. $\therefore$ the locus of X is the image of the locus of H, namely a straight line through C.

121. Let PQ be drawn $\parallel$ to BC, to meet AB, AC in P, Q. BQ, CP meet in O; AO produced meets BC in R. Then, by Ceva, $\frac{\text{AP}}{\text{PB}} \cdot \frac{\text{BR}}{\text{RC}} \cdot \frac{\text{CQ}}{\text{QA}} = 1$. But $\frac{\text{AP}}{\text{PB}} = \frac{\text{AQ}}{\text{QC}}$, $\therefore$ BR = RC; and the locus of O is the median through A.

122. Let $\angle$ APB be acute. Draw AD $\perp$ to BC. Then
$$\text{AB}^2 = \text{AP}^2 + \text{BP}^2 - 2\text{BP} \cdot \text{DP}, \quad \text{AC}^2 = \text{AP}^2 + \text{CP}^2 + 2\text{PC} \cdot \text{DP}.$$
$$\therefore m \cdot \text{AB}^2 + n \cdot \text{AC}^2 = (m+n)\,\text{AP}^2 + m \cdot \text{BP}^2 + n \cdot \text{CP}^2$$
$$= (m+n)\,\text{AP}^2 + n\text{PC} \cdot \text{BP} + m\text{BP} \cdot \text{PC}$$
$$= (m+n)\,(\text{AP}^2 + \text{BP} \cdot \text{PC}).$$

√

123. $\angle SAF = 90° - \angle ASy = 90° - \angle C$. Also $\angle AFE = \angle C$ ($\because$ F, E, C, B are concyclic), $\therefore$ AS is $\perp$ to FE.

124. (i) *Given two sides and the included median.* Construct a parallelogram having the two given sides as sides, and twice the median as diagonal; hence etc.

Given two sides, and the median bisecting one of them. Construct a $\triangle$ having for its sides the median, half the bisected side, and the whole other side; hence etc.

(ii) *Given a side* BC, *and the two medians through its extremities.* BG, CG are $\frac{2}{3}$ of the given medians; therefore $\triangle$ BCG can be constructed; hence etc.

Given two medians Bβ, Cγ *and the side* AC *bisected by* Bβ. The sides of $\triangle$ CGβ are known.

(iii) *Given three medians* Aα, Bβ, Cγ. If AG be produced to X, so that GX = AG; then BGCX is a parallelogram, of which the sides and a diagonal GX are known.

(iv) *Given* BC, AB ~ AC, $\angle C \sim \angle B$. Let AB > AC. From AB cut off AD = AC. Let $\angle BCD = \theta$. Then

$$\angle B + \angle \theta = \angle ADC = \angle ACD = \angle C - \angle \theta.$$

$\therefore \angle \theta = \angle \frac{B - C}{2}$ is known. Thus we know two sides and an opposite $\angle$ of $\triangle$ BDC. The triangle can then be constructed; $\angle$ BDC must be obtuse ($\angle$ ADC is acute), $\therefore$ there is only one solution.

(va) *Given* BC, AB − AC, $\angle$ B. Then $\triangle$ BCD (see Ex. iv) can be constructed.

(vb) *Given* BC, AB + AC, $\angle$ B. Suppose BA produced to E, so that AE = AC. Then we know BE, BC, $\angle$ B, and can construct $\triangle$ BEC. Hence etc.

(vi$a$) *Given* BC, $\angle$ A, AB − AC. $\angle$ BDC (see Ex iv) $= 90° + \frac{A}{2}$.

$\therefore \triangle$ BDC can be constructed (in only one way, for $\angle$ BDC is obtuse).

(vi*b*) *Given* BC, $\angle$ A, AB + AC. Let AB > AC. $\angle$ BEC (see Ex. v*b*) $= \frac{A}{2}$. $\therefore$ $\triangle$ BEC can be constructed (one solution only as $\angle$ BCE is obtuse).

(vii) *Given* D, E, F, *the feet of the* $\perp$*s.* Then A, B, C are the ex-centres of $\triangle$ DEF.

(viii*a*) *Given* $\angle$ B, *the altitude* AD, *and the perimeter.* Construct $\triangle$ ABD. This gives AB and therefore AC + CB. Along BD cut off BE = AC + CB; then the $\perp$ bisector of AE cuts BD at C.

(viii*b*) *Given* $\angle$ A, AD, *and the perimeter.* The solution depends on Theorem 13, i. Construct $\angle$ A, and cut off from its arms lengths AY_1, AZ_1 each = the semi-perimeter. Describe a circle to touch the arms at Y_1, Z_1. With centre A and radius AD describe a $\odot$. Then the base of the $\triangle$ will be an internal common tangent to these two $\odot$s.

(ix) *Given* $\angle$ B, *side* BC, *and* BQ *the bisector of* $\angle$ B. $\triangle$ BCQ can be constructed; hence $\triangle$ ABC.

(x) *Given* AB + AC, *and the angles.* Construct $\triangle A'B'C'$ having the given angles. Then make AB so that $AB : A'B' = AB + AC : A'B' + A'C'$. Hence $\triangle$ ABC.

(xi) See (x).

(xii*a*) *Given* $\angle$ A, *its bisector* AP, *and the altitude* AD. The rt. angled $\triangle$ ADP can be constructed. Draw lines making $\angle$s $\frac{A}{2}$ with AP on the two sides of it. These lines meet PD produced in B and C.

(xii*b*) *Given* $\angle$ A, *its bisector* AP, *and the altitude* BE. Construct $\triangle$ ABE, draw bisector AP, join BP and produce to meet AE in C.

(xiii) Let AD be the given altitude. Make $\angle$ DAB = 90° − B, $\angle$ DAC = 90° − C; this gives B and C. If either B or C is obtuse, the complement is negative, and AB, AC fall on the same side of AD.

(xiv) Let BA be produced to E, so that AE = AC. Let $\angle BCE = \theta$. Then $\angle B + \theta = 180° - \angle E$, and

$$\theta - \angle C = \angle ACE = \angle E, \quad \therefore\ 2\theta + \angle(B - C) = 180°.$$

$$\therefore\ \theta = 90° - \frac{B - C}{2},$$ a known angle. Hence $\triangle BCE$ can be constructed; and in only *one* way, $\because BE > BC$.

125. Let X be the image of H, the orthocentre, in the given side. Then X is on the circumcircle, which can now be drawn, cutting the side in two angular points, and HX in the third.

126. Draw the circumcircle, and let AH meet it in X. The $\perp$ bisector of HX cuts the $\odot$ in B and C.

127. Trisect the given median $B\beta$ at G ($BG = 2G\beta$). On $B\beta$ describe an arc containing $\angle A$. With G as centre and $\frac{2}{3}A\alpha$ as radius describe circle, cutting arc in A. Hence $\triangle ABC$.

128. $\triangle AEF$ is similar to $\triangle ABC$.

129. $BX^2 - CX^2 = BO^2 - CO^2$, etc. $\therefore\ \Sigma BX^2 = \Sigma CX^2$.

Converse theorem. If points X, Y, Z be taken on the sides of $\triangle ABC$, so that $BX^2 + CY^2 + AZ^2 = CX^2 + AY^2 + BZ^2$, the perpendiculars erected at these points are concurrent.

Proof. Let the $\perp$s at Y, Z meet at O; and draw $OX' \perp$ to BC. Then we find that $BX'^2 - CX'^2 = BX^2 - CX^2$.

$$\therefore\ BC(BX' - CX') = BC(BX - CX).$$

$\therefore$ X′ coincides with X.

130. $BX = s - c$, $CX = s - b$, etc. $\therefore$ the condition of Ex. 129 is fulfilled.

131. (i) K is the mid-point of arc BC, $\therefore\ KB = KC$. Again

$$\angle KBI = \angle KBC + \angle CBI = \angle KAC + \angle CBI = \frac{A}{2} + \frac{B}{2}.$$

And $\angle BKI = C$, $\therefore\ \angle BIK = 180° - C - \frac{A + B}{2} = \frac{A + B}{2} = \angle KBI$.

$$\therefore\ KI = KB.$$

(ii) $\angle CK'K = \angle YAI$ and $\angle K'CK = 90° = \angle AYI$,
$\therefore$ $\triangle$s $K'KC$, AIY are similar. $\therefore$ $KK' : CK = IA : YI$.
$\therefore$ $IA \cdot CK = KK' \cdot YI$. $\therefore$ $IA \cdot IK = 2Rr$.

132. If PIP' be the chord of the circumcircle that is bisected at I, $IA \cdot IK = IP^2 = R^2 - SI^2$, $\therefore$ by Ex. 31, $SI^2 = R^2 - 2Rr$.

133. $\triangle$s CDE, CAB are similar; and the ratio of the corresponding sides is constant, as $CD : CA = \cos C$. The distance of the centre of $\odot CDE$ from DE is therefore in a constant ratio to that of the centre of $\odot CAB$ from AB. But this last centre is a fixed point. $\therefore$ etc.

134. The projection of AE on HA (produced) = that of AB on $AC = AC = HA$. $\therefore$ KA produced bisects HE. Similarly, HA produced bisects EK. $\therefore$ etc.

135. The theorem is not true without restrictions.

To obtain the centres of the touching circles, the bisectors (internal or external) of the angles of the quadrilateral $ABCD$ must be drawn. It is necessary for the truth of the theorem that only one bisector of each angle shall be used.

Assuming $ABCD$ to be convex, the bisectors must be either all external or all internal. In either case it is easy to prove that a cyclic quadrilateral $PQRS$ is formed.

If the bisectors are internal $\angle P = \frac{A+B}{2}$, or $180° - \frac{A+B}{2}$,

$\angle R = \frac{C+D}{2}$, or $180° - \frac{C+D}{2}$.

$\therefore$ $\angle P + \angle R = \frac{A+B+C+D}{2}$ $(= 180°)$,

or $360° - \frac{A+B+C+D}{2}$ $(= 180°)$.

If the bisectors are external,

$$\angle P = \frac{A+B}{2}, \quad \angle R = \frac{C+D}{2}. \quad \therefore \quad \angle P + \angle R = 180°.$$

136. SD is $\parallel$ to AH, and $= \frac{AH}{2}$. $\therefore$ AS meets HD at V, where AV = 2AS; in fact, on the circumcircle. If HD also meets the circumcircle at U, $\angle AUV = 90^\circ$, and AD subtends a rt. $\angle$ at U. $\therefore$ etc.

137. D is the orthocentre of $\triangle BCF$, $\therefore \angle CPB = 90^\circ$, $\therefore$ locus of P is a $\odot$ on diameter BC.

138. (*a*) Let XY be drawn between O and the centre I. Then we have the ex-circle of $\triangle OPQ$; and semi-perimeter = OP (Theorem 13). Also

$$\angle XIY = 180^\circ - \angle IXY - \angle IYX = 90^\circ - \frac{POQ}{2}.$$

(*b*) Let XY be drawn not between O and I. Then (i) is untrue.

$$\angle XIY = 180^\circ - \frac{OXY}{2} - \frac{OYX}{2} = 90^\circ + \frac{POQ}{2}.$$

139. (i) $\angle BAH = 90^\circ - B$; $\angle BAS = 90^\circ - \angle ASY = 90^\circ - C$. $\therefore \angle SAH = B \sim C$.

(ii) H and N being given, we have S and the circumcircle; also $\angle SAH$. On SH construct an arc to contain this angle; this arc will cut the circumcircle in A (possibly 2 solutions). Where AH cuts the 9-pts. $\odot$ is D; hence B, C, where the $\perp$ to AD through D cuts the circumcircle.

As to the ambiguity, the $\odot$s of course cut in 2 points, but the angle subtended may be the supplement of the $\angle SAH$. The $\odot$ on SH is divided into a major and a minor arc by SH; a $\odot$ whose centre is S can either (i) cut both major and minor arcs. This will happen if H is outside the circumcircle; in this case only one intersection gives the correct angle. (ii) It may cut the minor arc in two points; this if H is inside the circumcircle. In this case there are 2 or 0 solutions according as $\angle SAH$ is acute or obtuse. (iii) It may fail to cut either arc.

140. See Ex. 103.

141. Draw the figure for the case in which $\angle B > \angle C$, and $\odot BIC$ cuts AB produced, and AC not produced.

Since I, C, E, B are concyclic, $\therefore \angle ICE = \angle ABI = \angle IBC$. But $\angle IBC = \angle IFA$, and $\angle ICE = \angle IFE$ ($\because$ I, C, F, E, are concyclic). $\therefore$ IF bisects $\angle AFE$. Hence I is the in-centre for $\triangle AFE$, and EF touches the in-circle of $\triangle ABC$.

142. The circumcentre of $\triangle HBC$ is the image of S in BC. If the three circumcentres are X, Y, Z, then $\triangle$s XYZ, $\alpha\beta\gamma$ have S as centre of similarity; $\triangle XYZ$ being on twice the scale. But $\triangle\alpha\beta\gamma$ is similar to $\triangle ABC$, on half the scale. $\therefore \triangle XYZ \equiv \triangle ABC$.

143. Let CD be the diameter through C, S the circumcentre of $\triangle BPQ$, BN the $\perp$ from B to CP. Then

$$\angle DBP = \angle DCP = \angle NBQ$$

($\because$ CD, CP are respectively $\perp$ to BQ, BN). Now BS makes with BP the same angle as BN with BQ. $\therefore$ S lies on BD.

144. $\triangle ADE$ is similar to $\triangle ABC$, and of 3 times the scale. The $\odot$ is the in-circle of $\triangle ADE$, $\therefore AF = 3s - 3a$. Also $AF = s$, $\therefore 2s = 3a$, $\therefore 2AF = 6s - 6a = 9a - 6a = 3a = DE$.

145. Let O be in-centre of $\triangle ABC$.

$$\angle AOE + \angle OEF = \frac{B}{2} + \frac{A}{2} + \angle BCF = \frac{B}{2} + \frac{A}{2} + \frac{C}{2} = 90^\circ.$$

$\therefore$ AD is $\perp$ to EF.

146. Let the vertices of the equilateral $\triangle$s be D, E, F. Then $\triangle$s AFC, ABE are obviously congruent, $\therefore CF = EB$. Hence $AD = BE = CF$. Also $\angle AFC = \angle ABE$, and if FC, EB meet in O, A, F, B, O are concyclic, $\therefore \angle BAO = \angle BFO$. Again, if AD be joined, $\triangle$s ABD, FBC are congruent, $\therefore \angle BAD = \angle BFO$. Hence $\angle BAO = \angle BAD$, and A, O, D are collinear, $\therefore$ AD, BE, CF are concurrent.

147. Using notation of Ex. 146, B, C, D, O are concyclic, and the circumcentre (or in-centre) of $\triangle$ BCD is on the $\perp$ bisectors of OB and OC. In fact, the $\triangle$ formed by the 3 circumcentres is identical with the $\triangle$ formed by the $\perp$ bisectors of OA, OB, OC. Now OA, OB, OC form 3 angles of 120°, $\therefore$ the said $\triangle$ is equiangular.

148. The required $\triangle$ is the pedal $\triangle$ of the $\triangle$ formed by the three given points.

149. Let AI_1 meet circumcircle of ABC in K. Then, as in Ex. 131, $KB = KC = KI_1$.

150. $\angle BIC = 180° - \frac{B}{2} - \frac{C}{2} = 90° + \frac{A}{2}$. Hence the base, altitude and vertical angle of $\triangle$ BIC are known and the $\triangle$ can be constructed. Hence A.

151. Common tangents cannot be drawn unless the triangle is obtuse-angled.

HS = 2HN; and radius of circumcircle = twice radius of 9-points $\odot$, $\therefore$ H is the external centre of similitude, and the intersection of the exterior common tangents.

152. Let α, β, γ be the mid-points of the sides; L the mid-point of AP. We will shew that AP makes with AB the same angle as Lα with AC; and that therefore the bisector of $\angle$ PLα is $\parallel$ to the bisector of $\angle$ BAC. Join $\alpha\gamma$, $\alpha\beta$, Lγ, Lβ. Draw XLY $\parallel$ to Aγ and $\beta\alpha$ cutting Aβ, $\gamma\alpha$ in X, Y, and ZLW $\parallel$ to Aβ and $\gamma\alpha$, cutting Aγ, $\beta\alpha$ in Z, W. Since BNMC are concyclic, $\angle$ ABP = $\angle$ ACP, $\therefore$ $\angle$ AγL = $\angle$ AβL. $\therefore$ $\triangle$s ZγL, XβL are similar, $\therefore$ LZ : AZ = LY : Yα. Hence $\angle$ LAZ = $\angle$ LαY. But αY is $\parallel$ to CA. $\therefore$ etc.

153. The orthocentre is a centre of similitude of the two circles (Ex. 151).

154. $\triangle\alpha\beta\gamma$ is similar and similarly situated to $\triangle$ABC. $\therefore$ if AP, BP, CP meet on the circumcircle of ABC, then the $\parallel$s through α, β, γ meet on the circumcircle of $\alpha\beta\gamma$; i.e. the 9-points $\odot$ of ABC.

155. Let the ⊥s be HM, HN. Then AH, MN are equally inclined to the bisector of ∠A, ∴ MN is ∥ to AS; and MN passes through P, the mid-point of AH. Also $AP = Sa$, ∴ PN passes through a.

156. (i) Let the centres of the circumscribed and inscribed ⊙s be S, I; the radii R, r. Then there is the relation $SI^2 = R^2 - 2Rr$ (Ex. 132). Let BC be *any* chord of the S ⊙ that touches the I ⊙. Construct a △ having base BC, in-radius r, and vertex on the S ⊙ and on the I side of BC; i.e. with given vertical ∠ (Ex. 150). There are two such △s, symmetrically disposed with respect to the S diameter bisecting BC. Let I′ be the in-centre of one of these △s. Then $SI'^2 = R^2 - 2Rr = SI^2$, ∴ I, I′ are equidistant from S. Also I, I′ are equidistant from BC, on the same side of it, ∴ one of the two positions of I′ must coincide with I; and we have constructed, with an arbitrary base, a △ inscribed to the S ⊙ and circumscribed to the I ⊙.

(ii) This depends on Feuerbach's Theorem that $NI = \frac{1}{2}R - r$. The locus of N is therefore a ⊙; and, S being fixed, the locus of H is a ⊙.

The following proof of Feuerbach's Theorem is due to V. Ramaswami Aiyar, M.A. (rearranged by Mr R. F. Davis).

AI meets circumcircle in K; KK′ is diameter. Bisect IK in L; complete ∥ogram LKaZ. Draw II′ ⊥ to KK′. N is mid-point of aP; on aK′ make $aN' = aN$. Then $aI' = r$, $oN' = \frac{1}{2}R$, $N'I' = \frac{1}{2}R - r$. Za (∥ to LK) is equally inclined to AH and AS, ∴ to N′a and Na. ∴ ZN′ = ZN, and, of course, ZI = ZI′. Also ∠ NaN′ = 2 ∠ ZaN′ = 2 ∠ IKI′ = ∠ ILI′. Again, $KI^2 = KC^2 = Ka \cdot KK'$. ∴ KK′ : KI = KI : K$a$. Now KK′ = 4$a$N. ∴ 4$a$N : 2$a$Z = 2LI : LZ. ∴ aN : aZ = LI : LZ. ∴ kites aNZN′, LIZI′ are similar. ∴ ∠ NZN′ = ∠ IZI′. ∴ ∠ N′ZI′ = ∠ NZI. ∴ △s N′ZI′, NZI are congruent. ∴ N′I′ = $\frac{1}{2}R - r$ = NI.

157. As $\triangle$s ADB, CDA are similar, $OD : O'D = AB : CA$. Also $\angle ODO' = \frac{1}{2} \angle BDC = 90^\circ = \angle BAC$, $\therefore$ $\triangle$s ODO', BAC are similar.

158. DE is a chord of $\odot$ DEPC, diameter CP, subtending $\angle C$, $\therefore$ $DE = CP \sin C$. Similarly $EF = AP \sin A$. If $DE = EF$, $\frac{AP}{CP} = \frac{\sin C}{\sin A} = \frac{AB}{CB}$; $\therefore$ P lies on a certain Apollonius $\odot$, which cuts $\odot$ ABC in B and another point. (Theorem 37, p. 78.)

159. As in Ex. 158, $PL \cdot MN = PC \sin PCL \cdot PA \sin A$. Now

$$\sin PCL = \frac{PB}{2R}, \quad \therefore \quad PL \cdot MN = \frac{PA \cdot PB \cdot PC}{2R} \cdot \sin A, \quad \therefore \text{ etc.}$$

160. See fig. 25,

$\angle ANM - \angle APM = 90^\circ - \angle PAM = 90^\circ - \angle PBC = 90^\circ - 2 \angle CSP$, etc.

161. This is a particular case of Ex. 160. It may also be proved as follows. Let P, Q be opposite ends of a diameter; let X, p, q be the images of H, P, Q in BC. Then XP, XQ are at rt. $\angle$s. $\therefore$ Hp, Hq, the images of XP, XQ, are at rt. $\angle$s, $\therefore$ the Simson lines of P, Q are at rt. $\angle$s. (See Ex. 116.)

162. Let XY be any line $\perp$ to the median Aα, and cut AB, AC in X, Y. Draw XU, YU $\perp$ to AB, AC. Let AU meet the circumcircle in V; then V is the point required.

163. (i) If $\odot$s BRP, CPQ meet at O, then

$\angle QOR = 360^\circ - POR - POQ = B + C = 180^\circ - A$,

$\therefore$ A, Q, R, O are concyclic.

(ii) $\angle BOC = \angle POB + \angle POC = \angle PRB + \angle PQC = \angle A + \angle QPR$ = const.

$\therefore$ O is a point at which the sides of ABC subtend fixed angles, $\therefore$ a fixed point.

(iii) If ABC, PQR are similar, $\angle BOC = 2\angle A$, etc. $\therefore$ O is the circumcentre of ABC. Also

$$\angle OPR = \angle OBA = 90^\circ - \angle C = 90^\circ - \angle PRQ.$$

$\therefore$ PO is $\perp$ to QR, $\therefore$ etc. $\therefore$ O is the orthocentre of $\triangle$ PQR.

164. Let H be the orthocentre, S the circumcentre of $\triangle OAB$; P the mid-point of AB. Then

$$OH = 2SP = 2 \cdot \frac{AB}{2} \cdot \cot PSB = AB \cot AOB = \text{const.}$$

$\therefore$ locus of H is a $\odot$ round O.

165. The circumcircle of the $\triangle$ formed by the three lines. See Ex. 107.

166. The $\odot$s are two of the four $\odot$s touching the sides of the $\triangle$. Given a $\triangle$ PQR formed by three of the four centres of touching $\odot$s, the $\triangle$ ABC is the pedal $\triangle$ of $\triangle$ PQR. $\odot$ ABC is the 9-points $\odot$ of $\triangle$ PQR; and therefore $\odot$ ABC bisects the line joining any two centres of touching $\odot$s.

167. Let $\angle A = 45^\circ$. Then the 9-points centre lies on EF. For

$$\angle HDF = \angle HBA = 45^\circ \text{ and } \angle HDE = \angle HCA = 45^\circ.$$

$\therefore \angle EDF = 90^\circ$. $\therefore$ EF is a diameter of the 9-points $\odot$.

168. Let K, L be the mid-points of the arcs BC, BAC. Then (see Ex. 131), it is easily shewn that $KI \cdot KI_1 = KB^2$ = constant. $\therefore$ I, I_1 are extremities of a diameter of a fixed $\odot$ of centre K and radius KB. Similarly I_2, I_3 are extremities of a diameter of a fixed $\odot$ of centre L and radius LB.

169. Let AB be $>$ AC and have regard to sense of lengths along BC; draw AD $\perp$ to BC; $\dfrac{BY}{AB} = \dfrac{CY}{AC} = \dfrac{BC}{AB + AC}$.

$$AX^2 - AY^2 = DX^2 - DY^2 = (DX - DY)(DX + DY)$$
$$= (CX - CY)(DB + DC) = (YB - CY)(DB + DC)$$
$$= \frac{(AB - AC)\, BC\, (DB + DC)}{AB + AC}$$

$$= \frac{(AB - AC)(DB - DC)(DB + DC)}{(AB + AC)}$$
$$= \frac{AB - AC}{AB + AC} \cdot (DB^2 - DC^2)$$
$$= \frac{AB - AC}{AB + AC} (AB^2 - AC^2) = (AB - AC)^2.$$

170. Take p, q, X, the images of P, Q, H in AB. Let M, N be the projections of P, Q on AB. Then the Simson lines of P, Q are the ‖s through M, N to pH, qH. Now MN is = and ‖ to pq; ∴ the Simson lines meet on CH.

171. Same proof as for Simson line.

172. The θ line makes with BC an angle = ∠ BPN. Now ∠ BPN ~ ∠ BPN′ = $\theta \sim \theta'$, ∴ etc.

173. We must shew that the locus of Y, the projection of P on LMN, is a straight line. Now

$$\frac{PY}{PM} = \sin PMY = \sin PAB = \text{const.};$$

and PY makes a fixed angle with PM. ∴ the locus of Y is similar to the locus of M, namely a straight line. ∴ the envelope of LMN is a parabola with focus P.

As a particular case, M, N may occupy the positions C, B, since the angle between PC and CA = the angle between PB and AB. Thus BC is a tangent to the parabola; and similarly CA, AB.

The figure PMNL remains similar to itself, as all its angles remain fixed. It is a minimum when PL is a minimum; i.e. when PL is ⊥ to BC. We thus have Simson line as the tangent that is nearest to the focus; i.e. the tangent at the vertex of the parabola.

174. This follows immediately from $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

175. If BE is ⊥ to CA, then BE is a chord of the ⊙ on BC as diameter. Also BE subtends 60° at the circumference of this ⊙, and is a side of an equilateral △ inscribed therein.

176. The theorem is true only if ABC, A′B′C′ are in the same sense.

The two triangles are congruent; and the one may be made to coincide with the other by sliding it round the circumference. Hence AA′ = BB′ = CC′. Also AA′bc, BB′ca, CC′ab are cyclic quadrilaterals. And the three ⊙s are equal, as equal chords AA′, BB′, CC′ subtend equal angles at the circumferences. Now bc, ca, ab subtend angles A, B, C, at the circumferences of their respective (equal) ⊙s. ∴ $bc : ca : ab$ = BC : CA : AB.

If the triangles are in opposite senses, they have an axis of symmetry, on which lie a, b, c.

177. The angle between the two tangents from a point to a fixed ⊙ being given, the lengths of the tangents are determined. Thus the sides of the triangle are determined.

178. ∠ (BC, B′C′) = ∠ (CA, C′A′) = ∠ (AB, A′B′). ∴ ∠ between tangents from a = ∠ between tangents from b = ∠ between tangents from C, ∴ a, b, c are equidistant from centre.

179. ∠ PCM = ∠ PBN = ∠ PLY, ∴ △s PCM, PLY are similar. ∴ PC : PM = PL : PY, ∴ PL . PM = PC . PY.

CHAPTER V.

180. No; for X, X′ might be internal and external points of division.

181. (i) $\frac{B\alpha}{C\alpha} \cdot \frac{C\beta}{A\beta} \cdot \frac{A\gamma}{B\gamma} = (-1)(-1)(-1) = -1$. ∴ A$\alpha$, B$\beta$, C$\gamma$ are concurrent. (Converse of Ceva.)

(ii) If AX, BY, CZ are the bisectors of ∠s,

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = \left(-\frac{b}{c}\right)\left(-\frac{c}{a}\right)\left(-\frac{a}{b}\right) = -1, \text{ ∴ etc.}$$

(iii) $\triangle$s AEF, ABC being similar, $\frac{AE}{AF} = \frac{c}{b}$.

$$\therefore \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{AF}{AE} \cdot \frac{BD}{BF} \cdot \frac{CE}{CD} = \frac{b}{c} \cdot \frac{c}{a} \cdot \frac{a}{b}$$

(numerically) $= 1$. By drawing figures of acute and obtuse angled $\triangle$s, it is seen that the product $= -1$, $\therefore$ etc.

182. Let the join of A to the meet of BY, CZ cut BC in X. Then $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1$ (Ceva). $\therefore$ BX $= -$ CX, $\therefore$ X is the mid-point of BC.

183. $BX = X'C$, $\therefore$ $CX = X'B$, $\therefore$ $\frac{BX'}{CX'} = \frac{BX}{CX}$. Now

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1 \text{ (Ceva)}, \quad \therefore \frac{BX'}{CX'} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'} = -1.$$

$\therefore$ AX′, BY′, CZ′ are concurrent (converse of Ceva).

184. Here $BX = s - b$, $CX = s - c$, etc. numerically.

$$\therefore \frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1 \text{ (numerically)}$$

$= -1$ (by inspection), $\therefore$ etc.

185. $BX_1 = s - c$, $CX_1 = s - b$, etc. numerically.

$$\therefore \frac{BX_1}{CX_1} \cdot \frac{CY_1}{AY_1} \cdot \frac{AZ_1}{BZ_1} = \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} \cdot \frac{s-b}{s-a} = 1 \text{ (numerically)},$$

$\therefore$ etc.

186. Let MN cut BC in L′. Then, as in Theorem 22,

$$\frac{BL'}{CL'} = \frac{BL}{CL}, \quad \therefore \text{ L, L′ coincide.}$$

187. $AN = AP\cos\theta$, $BN = BP\cos(C - \theta)$, $BL = -BP\cos(A + \theta)$, $CL = CP\cos\theta$, $CM = CP\cos(C - \theta)$, $AM = -AP\cos(A + \theta)$, $\therefore$ etc.

188. $\frac{BP}{CP} \cdot \frac{CR}{AR} \cdot \frac{AQ}{BQ} = 1$, $\therefore$ $\frac{BP}{CP} = \frac{1}{4}$.

189. $\frac{CQ}{AQ} = -\frac{a}{c}$, $\frac{AR}{BR} = -\frac{b}{a}$, $\therefore \frac{BP}{CP} \cdot \frac{a}{c} \cdot \frac{b}{a} = 1$,

$\therefore \frac{BP}{CP} = \frac{c}{b}$, $\therefore$ AP is the external bisector of $\angle$ A.

190. Consider QPC cutting sides of $\triangle$ ABa. Then

$$\frac{BC}{aC} \cdot \frac{aP}{AP} \cdot \frac{AQ}{BQ} = 1.$$

But $\frac{BC}{aC} = 2$, $\frac{aP}{AP} = -1$, $\therefore \frac{AQ}{BQ} = -\frac{1}{2}$, $\therefore AQ = \frac{1}{3}AB$.

191. $\frac{BP}{CP} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = -1$, $\frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN} = +1$, $\therefore \frac{BP}{CP} = -\frac{BL}{CL}$.

192. $\frac{BA''}{CA''} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1$, $\therefore \frac{BA''}{CA''} = \frac{B'A}{CB'} \cdot \frac{C'B}{AC'} = \frac{CA'^2}{A'B^2}$.

193. Let OL, AP meet in X; KP, LP′ in Y; AP, LK in V. Considering $\triangle$ LOP′, we must shew that $\frac{OB}{P'B} \cdot \frac{P'Y}{LY} \cdot \frac{LX}{OX} = 1$.

$\because$ LK is $\parallel$ to P′P, $\therefore \frac{P'Y}{LY} = \frac{P'P}{LK} = \frac{2OP}{OB}$; and

$$\frac{LX}{OX} = \frac{LV}{OP} = \frac{CP}{2OP} = \frac{P'B}{2OP}.$$

$\therefore$ the product of the three ratios is 1.

Aliter. Let KP and OL cut at Q. Apply Menelaus to $\triangle$ OPQ, transversal LP′Y; and apply the converse to $\triangle$ OPQ and so prove BXY a straight line.

194. Let AI cut BC at V. Then by Ceva and Menelaus

BU : CU = − BV : CV = − AB : AC.

$\therefore$ AU is external bisector of $\angle$ A, $\therefore$ etc.

195. Since F, E, X are collinear, $\frac{BX}{CX} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1$.

$$\therefore \frac{BX}{CX} = \frac{AE}{CE} \cdot \frac{BF}{AF} = \frac{s-a}{s-c} \cdot \frac{s-b}{s-a} = \frac{s-b}{s-c}, \quad \therefore \Pi \frac{BX}{CX} = 1,$$

$\therefore$ X, Y, Z are collinear.

196. Consider $\triangle AQR$, and the transversal BCP.

$$\frac{QP}{RP}\cdot\frac{RB}{AB}\cdot\frac{AC}{QC}=1,\ \therefore\ \frac{PQ}{PR}=\frac{AB}{AC}.$$

197. $AE\,.\,AE'=AF\,.\,AF',\ \therefore\ \frac{AE}{AF}=\frac{AF'}{AE'}$. Now

$$\frac{AE}{AF}\cdot\frac{BF}{BD}\cdot\frac{CD}{CE}=-1,\quad\therefore\ \frac{AF'}{AE'}\cdot\frac{BD'}{BF'}\cdot\frac{CE'}{CD'}=-1,$$

$\therefore$ AD′, BE′, CF′ are concurrent.

198. Consider $\triangle DCB$. Let BH meet CD in V, DG meet CB in U.

$$\frac{BF}{DF}\cdot\frac{DV}{CV}\cdot\frac{CU}{BU}=\frac{FE}{FH}\cdot\frac{FH}{EH}\cdot\frac{KG}{FG}=-1,\ \therefore\ \text{etc.}$$

199. Let the $\perp$s to BC, AC be PM, PN.

By Menelaus, $\frac{AQ}{BQ}\cdot\frac{BM}{CM}\cdot\frac{CN}{AN}=1.$

$$\therefore\ \frac{PQ+AP}{PQ-PB}\cdot\frac{BP^2}{AP^2}=1.$$

$$\therefore\ PQ\,(AP^2-BP^2)=AP\,.\,BP\,(BP-AP).$$

$$\therefore\ PQ\,(PA+PB)=PA\,.\,PB.$$

$$\therefore\ 2PO\,.\,PQ=PA\,.\,PB.$$

200. Let YZ, FE meet at P, and let AD meet EF in Q. Consider $\triangle DEF$. The straight line YZ meets DE, DF in Z, Y; and the join of D to the intersection of EY, FZ meets EF in B. $\therefore$ by Ex. 191, EF is divided in the same ratio at P and Q. Now DQ bisects $\angle EDF$, $\therefore$ DE : DF = EQ : FQ = EP : FP. $\therefore$ P lies on the exterior bisector of $\angle EDF$, i.e. on BC.

201. Let X, Y be the mid-points of PQ, PR. Since T, X, S are collinear, $\therefore\ \frac{QS}{RS}\cdot\frac{RT}{PT}\cdot\frac{PX}{QX}=1.$

$\therefore\ \frac{RT}{PT}=-\frac{RS}{QS}$. Again, as U, Y, S are collinear

$$\frac{QS}{RS}\cdot\frac{RY}{PY}\cdot\frac{PU}{QU}=1,\ \therefore\ \frac{PU}{QU}=-\frac{RS}{QS}.$$

Lastly, as T, U, V are collinear

$$\frac{QV}{RV}\cdot\frac{RT}{PT}\cdot\frac{PU}{QU}=1,\quad\therefore\ \frac{QV}{RV}=\frac{QS^2}{RS^2}.$$

202. Let ZY_1 meet BC in P; consider $\triangle AZY_1$ and transversal BPC. Then $\frac{ZP}{Y_1P} \cdot \frac{Y_1C}{AC} \cdot \frac{AB}{ZB} = 1$. But $Y_1C = ZB = s - b$, $\therefore$ ZY_1 is divided in the ratio $AC : AB$.

203. By similar $\triangle$s, $DH = EC \cdot \frac{DG}{EG}$.

From $\triangle DEC$ and transversal BGF,

$$\frac{DG}{EG} \cdot \frac{EB}{CB} \cdot \frac{CF}{DF} = 1, \quad \therefore \quad \frac{DG}{EG} = \frac{DF}{CF} \cdot \frac{CB}{EB},$$

$$\therefore \quad DH = DF \cdot \frac{EC}{CF} \cdot \frac{CB}{EB} = DF \cdot \frac{EB}{BA} \cdot \frac{CB}{EB} = DF.$$

204. In $\triangle FGH$

$$\frac{FE}{FA} \cdot \frac{HA}{HB} \cdot \frac{GB}{GE} = \frac{DC}{DA} \cdot \frac{AD}{BC} \cdot \frac{CB}{CD} = -1, \quad \therefore \text{ etc.}$$

205. Let AD, BD be $\parallel$ to CB, CA; AD, CF meet in P; BD, CF in Q, area of $\triangle ABC = \triangle$. Let $BC = a$, $AC = b$; then by similar $\triangle$s $PA = na$, $\therefore$ $DP = a(1 - n)$. Similarly

$$DQ = BQ - BD = b\left(\frac{1-n}{n}\right).$$

Now $\angle C + \angle PDQ = 180°$.

$$\therefore \quad \triangle DPQ : \triangle ABC = DQ \cdot DP : CA \cdot CB$$

$$= \left(\frac{1-n}{n}\right)^2 ab : ab = (1-n)^2 : n.$$

206. $\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = \frac{\triangle BOC}{\triangle ABC} + \frac{\triangle COA}{\triangle ABC} + \frac{\triangle AOB}{\triangle ABC} = 1.$

CHAPTER VI.

207. $\frac{AC}{CB} \Big/ \frac{AD}{DB} = -1, \quad \therefore \quad AD = 2BD = 2(AD - AB), \quad \therefore \quad AD = 12$ cm.

208. (i) $AC = 2$ cm., $AD = -6$ cm.
(ii) $AC = 12$ cm., $AD = 4$ cm.
(iii) $AC = -6$ cm., $AD = 2$ cm.

209. $\frac{AC}{CB} \Big/ \frac{AD}{DB} = -1, \quad \therefore \quad \frac{CA}{AD} \Big/ \frac{CB}{BD} = -1.$

210. Ratio $= -1$. Yes.

211. $\triangle ABC$; AD, AE the bisectors of $\angle A$. $\frac{BD}{DC} - \frac{AB}{AC}$ and is positive, $\frac{BE}{EC} = \frac{AB}{AC}$ and is negative, $\therefore \frac{BD}{DC} \Big/ \frac{BE}{EC} = -1$.

212. Let $BA = x$, $BC = y$, $BD = z$. Then $\frac{y - x}{y} \Big/ \frac{z - x}{z} = -1$,

$\therefore$ (as in Theorem 25) $\frac{1}{y} + \frac{1}{z} = \frac{2}{x}$.

Let $CA = x$, $CB = y$, $CD = z$. Then $\frac{-x}{y} \Big/ \frac{z - x}{y - x} = -1$,

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{2}{z}.$$

Let $DA = x$, etc.

213. Let $OB = b$, $OC = c$, $OD = d$; then $AO = b$ and $b^2 = cd$.

$$\{AB, CD\} = \frac{AC}{CB} \Big/ \frac{AD}{DB} = \frac{b + c}{b - c} \Big/ \frac{b + d}{b - d} = \frac{b^2 + bc - bd - cd}{b^2 + bd - bc - cd}$$

$$= \frac{bc - bd}{bd - bc} (\because b^2 = cd), \quad \therefore \{AB, CD\} = -1.$$

214. Let $PA = a$, $PB = b$, $PC = c$, then $DP = c$.

$$\{AB, CD\} \equiv \frac{AC}{CB} \Big/ \frac{AD}{DB} = -1, \quad \therefore \frac{c - a}{b - c} \Big/ \frac{-c - a}{b + c} = -1,$$

$\therefore$ etc., $\therefore c^2 = ab$.

215. O is mid-point of AB,

$\therefore OC \cdot OD = OB^2$, $\therefore (OP - CP)(OP + PD) = OB^2$,

but $CP = PD$, $\therefore OP^2 - CP^2 = OB^2$.

216. $OC \cdot OD = OB^2 = OA^2$, $\therefore$ D coincides with (i) A, (ii) the pt. at infinity, (iii) B, (iv) O.

217. $OA \cdot OB = OC^2$, $\therefore \frac{OA}{OC} = \frac{OC}{OB} = \frac{OC - OA}{OB - OC} = \frac{AC}{CB}$.

218. Let O be the centre of the circle.

Then $OP \cdot OQ = (\text{radius})^2 = OP' \cdot OQ'$,

$\therefore$ P, P', Q, Q' are concyclic.

219. By Menelaus' Theorem $\frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$,

$$\therefore \frac{BX'}{CX'}\left(-\frac{s-c}{s-a}\right)\left(-\frac{s-a}{s-b}\right) = 1, \quad \therefore \frac{BX'}{X'C} = -\frac{s-b}{s-c},$$

$$\therefore \{BC, XX'\} = \frac{BX}{XC} \Big/ \frac{BX'}{X'C} = \frac{s-b}{s-c} \Big/ \left(-\frac{s-b}{s-c}\right) = -1.$$

220. Let X, Y, Z be the points of contact of the excircle opposite A. Then by Menelaus' Theorem

$$\frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1, \quad \therefore \frac{BX'}{CX'} \cdot \frac{s-b}{s} \cdot \frac{s}{s-c} = 1,$$

$$\therefore \frac{BX'}{X'C} = -\frac{s-c}{s-b},$$

$$\therefore \{BC, XX'\} = \frac{BX}{XC} \Big/ \frac{BX'}{X'C} = \frac{s-c}{s-b} \Big/ \left(-\frac{s-c}{s-b}\right) = -1.$$

221. $\{AB, CD\} = -1$; $PR/RQ = 1$. See Theorem 27 (first part).

222, 223, 225. See Theorem 27.

226. The parallel to the base through that vertex.

227. $\beta\gamma$ is $\parallel$ to the arm αC, and is bisected by αA.

$$\therefore \alpha\{\gamma\beta, AC\} = -1.$$

228. DA, DB are the internal and external bisectors of $\angle EDF$.

$$\therefore D\{EF, AB\} = -1.$$

229. Produce YZ to cut BC produced at X'. Let AX, YZ intersect at W.

Then $A\{BC, XX'\} = -1$, as in Ex. 219.

$$\therefore \{ZY, WX'\} = -1, \quad \therefore X\{ZY, AX'\} = -1.$$

230. Let AB be a side of the parallelogram, E the mid-pt. of AB and O the intersection of the diagonals.

Then AB is parallel to an arm of the pencil to be proved harmonic; and OA, OE, OB are the remaining arms of the pencil and $AE = EB$.

$\therefore$ the pencil is harmonic.

231. Let OD (produced if necessary) cut A′B′C′D′ in D″.
Now $\{AB, CD\} = -1$. $\therefore$ O $\{AB, CD\} = -1$.
$\therefore$ $\{A'B', C'D''\} = -1$.
But $\{A'B', C'D'\} = -1$.
$\therefore$ D′ and D″ coincide. $\therefore$ O, D′, D″ are collinear.

232. (i) Suppose none of the points A, B, C, D lie on the arc between O and P. Then $\angle AOC = \angle APC$, etc., etc.
$\therefore$ the pencils O $\{AB, CD\}$, P $\{AB, CD\}$ are congruent.
(ii) Suppose P is on the arc AD and O on the arc BD.
Produce DO to D′, then the pencils O $\{AB, CD'\}$, P $\{AB, CD\}$ are congruent, etc., etc.
$\therefore$ if O $\{AB, CD\} = -1$, then P $\{AB, CD\} = -1$.

233. Draw BY, CZ $\perp$ to AX; and let the external bisector of $\angle$ A cut BC produced at X′ (then $\angle XAX'$ is a rt. $\angle$).
Let θ be the acute $\angle$ between AX and BC.

$$\text{Now } \{BC, XX'\} = -1, \text{ i.e. } \frac{BX}{XC} \Big/ \frac{BX'}{X'C} = -1,$$

$$\therefore \quad \frac{BX \cos\theta}{XC \cos\theta} \Big/ \frac{BX' \cos\theta}{X'C \cos\theta} = -1,$$

$$\text{i.e. } \frac{YX}{XZ} \Big/ \frac{YA}{AZ} = -1,$$

$$\text{i.e. } \{YZ, XA\} = -1.$$

$$\therefore \{AX, YZ\} = -1.$$

234. ABCD is a square, P any pt. on the circumscribing circle.
Suppose P to be on the arc AB.
Since arc AD = arc DC, $\therefore$ PD is the internal bisector of $\angle$ APC.
Since BD is a diameter, $\therefore$ $\angle$ DPB is a rt. $\angle$.
$\therefore$ PB is the external bisector of $\angle$ APC.
$\therefore$ P $\{AC, DB\} = -1$. Th. 28.

235. Since the diameter CD cuts the chord AB at rt. $\angle$s,
$\therefore$ arc AD = arc BD, $\therefore$ PD is either the internal or external bisector of $\angle$ APB. Also $\angle CPD = 90°$.
$\therefore$ PC, PD are the bisectors of $\angle$ APB.
$\therefore$ P $\{AB, CD\} = -1$.

236. Since $\beta\gamma$ is $\parallel$ to BC and $Ba = aC$, $\therefore\ \gamma X = X\beta$.
$\therefore\ a\{\gamma\beta, XC\} = -1, \quad \therefore\ a\{YZ, XW\} = -1.$

237. Let PA, PB, PC be the given lines.
Draw PD so that $P\{AB, CD\} = -1$; through the given point draw a parallel to PD; this is the required line (by the first part of the proof of Theorem 27).

238. Let A, B, C be the three given points. We want to find P so that PA, PB, PC and the given line may form a harmonic pencil.
Let AB cut the given line at D; in ABD find a point C′ such that A, B, C′, D is a harmonic range. Join CC′ and let it cut the given line at P.
Then PA, PB, PC, PD form a harmonic pencil.
$\therefore$ a parallel to PD is divided into two equal parts by PA, PB, PC.

239. By Ceva's Theorem $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1.$
By Menelaus' Theorem $\frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1,$
$$\therefore\ \frac{BX}{XC} \Big/ \frac{BX'}{X'C} = -1.$$

240. $\angle PQR$ is a rt. $\angle$, $\therefore$ we want to prove that QP bisects $\angle TQN$. Now $\angle NQP = 90° - \angle NPQ = \angle QPT = \angle TQP$.

241. Draw AE the external bisector of $\angle A$ and let it cut BC produced at E. Bisect DE at P.
Then $\{BC, DE\} = -1$, and P is mid-point of DE.
$$\therefore\ PB \cdot PC = PD^2.$$

242. $2\frac{PA}{AC} = 2\frac{PC + CA}{AC} = 2\frac{PC}{AC} - 2;$
and $$\frac{PB}{BC} + \frac{PD}{DC} = \frac{PC + CB}{BC} + \frac{PC + CD}{DC}$$
$$= PC\left(\frac{1}{BC} + \frac{1}{DC}\right) - 2.$$
But, since $\{AB, CD\} = -1$, $\frac{2}{AC} = \frac{1}{BC} + \frac{1}{DC}$.

243. Describe a pair of intersecting circles, the one to go through A and B, and the other through C and D. Let X, Y be their points of intersection; let XY cut the given line at Z. From Z draw ZT a tangent to one of the circles. From the given line cut off ZP and ZQ each equal to ZT. Then P and Q are the required points.

For Z is the mid-pt. of PQ and

$$ZP^2 = ZT^2 = ZX \,.\, ZY = ZA \,.\, ZB,$$

and also $= ZC \,.\, ZD$.

[Note that this solution breaks down unless Z is outside the circles, i.e. unless C and D both lie between A and B or neither lie between A and B. It can be seen that P and Q are imaginary unless these conditions obtain.]

244. BDC is $\parallel$ to SRA,

$\therefore$ by similar $\triangle$s, $AR : CD = QR : QD = RS : DB$.

But $CD = DB$, $\therefore AR = RS$.

Or thus: Draw QX to BC and SA; then QX cuts BC and SA at infinity. Since $BD = DC$,

$\therefore Q\{BC, D\infty\} = -1$, $\therefore Q\{SA, R\infty\} = -1$, $\therefore AR = RS$.

245. [$S\{PQ, AB\} = -1$, $\therefore$ we have only to prove TQ $\parallel$ to SB.]

Since AB is a diameter, $\therefore \angle ATB$ is a rt. $\angle$, $\therefore$ by Th. 29, $\angle QTB = \angle BTS = \angle TBS$ ($\because ST = SB$), $\therefore$ TQ is $\parallel$ to SB.

Now $S\{PQ, AB\} = -1$ and TQ is $\parallel$ to SB, $\therefore$ SA bisects TQ (by the first part of proof of Theorem 27).

246. $\triangle BOA : \triangle EOA = BA : AE = BC : OC$,

$$\therefore \frac{1}{\triangle BOA} = \frac{1}{\triangle EOA}\,\frac{OC}{BC}.$$

Similarly $$\frac{1}{\triangle COA} = \frac{1}{\triangle FOA}\,\frac{OB}{BC} = \frac{1}{\triangle EOA}\,\frac{OB}{BC},$$

$$\therefore \frac{1}{\triangle BOA} + \frac{1}{\triangle COA} = \frac{1}{\triangle EOA} = \frac{2}{\parallel^{\text{ogram}} OEAF}.$$

247. Let TQ, PR intersect at S. Since

$$TP = TQ, \ \angle TQP = \angle TPQ = \angle PQN \text{ (by } \parallel\text{s)}.$$

PR is a diameter, $\therefore \angle PQR = 90°$, $\therefore Q\{NS, PR\} = -1$.

$\therefore T\{NS, PR\} = -1$; and QN is $\parallel$ to TP, $\therefore$ TR bisects QN.

248. Let YZ cut AD at W and BC at V. By Ex. 239,
$\{BC, DV\} = -1$, $\therefore$ A $\{BC, DV\} = -1$, $\therefore$ $\{ZY, WV\} = -1$,
$\therefore$ D $\{ZY, WV\} = -1$; but $\angle WDV = 90°$, $\therefore$ DA bisects $\angle YDZ$.

249. Let AB be the fixed chord, CD the $\perp$ diameter, and P any pt. on the circle.
Now arc CB = arc CA, and arc DB = arc DA.
$\therefore$ PC, PD are the bisectors of $\angle APB$,
$\therefore$ P $\{CD, AB\} = -1$,
$\therefore$ etc.

250. By Ceva's Theorem $\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = -1$.
Again $\{BC, A'A''\} = -1$, $\therefore$ $BA'/A'C = -BA''/A''C$; etc.
$\therefore$ $\frac{BA''}{CA''} \cdot \frac{CB''}{AB''} \cdot \frac{AC''}{BC''} = +1$, $\therefore$ A'', B'', C'' are collinear (converse of Menelaus' Theorem).

251. Since VA', VB', VC' are the internal bisectors of $\angle$s BVC, CVA, AVB, $\therefore$ $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{VB}{VC} \cdot \frac{VC}{VA} \cdot \frac{VA}{VB} = 1$, and is positive.
Again $\{BC, A'A''\} = -1$, $\therefore$ $\frac{BA'}{A'C} = -\frac{BA''}{A''C} = \frac{BA''}{CA''}$, etc., etc.
$\therefore$ $\frac{BA''}{CA''} \cdot \frac{CB''}{AB''} \cdot \frac{AC''}{BC''} = +1$, $\therefore$ A'', B'', C'' are collinear (converse of Menelaus' Theorem).

252. Let BC cut $\odot$ B'C'B in P, $\odot$ B'C'C in Q, and B'C' in O.
Then OP . OB = OB' . OC' = OA'^2; similarly OQ . OC = OA'^2.
Also $\{OA', BC\}$ is harmonic,
$\therefore$ $\frac{1}{OB} - \frac{1}{OA'} = \frac{1}{OA'} - \frac{1}{OC}$.
Now $A'P = OP - OA' = \frac{OA'^2}{OB} - OA' = OA'^2\left(\frac{1}{OB} - \frac{1}{OA'}\right)$,
and $QA' = OA' - OQ = OA' - \frac{OA'^2}{OC} = OA'^2\left(\frac{1}{OA'} - \frac{1}{OC}\right)$,
$\therefore$ $A'P = QA'$.

CHAPTER VII.

253. Using the notation of fig. 33, first prove $\triangle CPT \equiv \triangle CQT$, then $\triangle CPN \equiv \triangle CQN$.

254. The tangent at the point.

255. Draw TP, TQ tangents from T; let PQ cut CT at N'. Then by Theorem 30, $CN' \cdot CT = CA^2$, $\therefore$ N and N' coincide; and by Ex. 253, PQ is $\perp$ to CT, $\therefore$ XY and PQ coincide.

256. The polar is (i) the tangent at T, (ii) the line at infinity, (iii) the line through C at right angles to CT.

257. Let C be the common centre; let T be any pt. on the circle B, and let CT cut the polar of T with respect to A in N.

Then $CN \cdot CT = (\text{radius of A})^2$.

$\therefore$ CN is const.; and the polar is $\perp$ to CN.

$\therefore$ the envelope of the polar is another concentric circle.

258. (i) The lines joining the points of contact.
(ii) The tangents to the circumcircle at the vertices.

259. Using the notation of Chap. IV. (see fig. 19),

$$AF \cdot AB = AE \cdot AC = AH \cdot AD.$$

$\therefore$ FC is the polar of B, EB is the polar of C, and H is the pole of BC.

260. $\angle XKT = \text{a rt. } \angle = \angle XNT$, $\therefore$ X, K, N, T are concyclic,
$\therefore$ $CN \cdot CT = CK \cdot CX = CS^2$ ($\because$ SR is the polar of X).

261. See Theorem 31.

262. Let the line cut the polar of T in H'; then the chord is divided harmonically at T and H', but it is also divided harmonically at T and H, $\therefore$ H and H' coincide, $\therefore$ H lies on the polar of T.

263. Let RS cut the polar of T in H. Then $N\{RS, HT\} = -1$, and $\angle TNH$ is a rt. $\angle$. $\therefore$ NH bisects $\angle RNS$ (Th. 29).

264. $TR \cdot TS = KT^2 - KR^2 = CT^2 - CR^2 = CT^2 - CN \cdot CT = CT \cdot NT$
$= TH \cdot TK$ (since K, H, N, C are concyclic).

265. The pencil subtended at A is $A\{CD, BO\}$ which is harmonic by Theorem 31.

Consider the pencil subtended at any pt. P on the circle; produce all the rays of the pencil through P.

Since the angles subtended at any point of a circle by a chord of the circle are equal or supplementary, the pencils $P\{CD, BA\}$, $A\{CD, BO\}$ can be made to coincide with one another. $\therefore$ $P\{CD, BA\}$ is harmonic.

266. Let C be the centre of the circle; and let PQ, CO intersect at T. Draw OR at rt. $\angle$s to CO.

Then OR, OT are the bisectors of $\angle$ POQ,

$\therefore$ $O\{PQ, RT\} = -1$, $\therefore$ OR is the polar of T.

But OR is a fixed line, $\therefore$ T is a fixed point.

If O approaches the centre along a definite diameter, T will move to infinity along that line and will in the limit coincide with the point at infinity on that line; unless the diameter is defined the theorem is meaningless.

267. [Misprint in first edition; for 'Ex. 344' read 'Ex. 266'.]

If a chord PQ of a circle moves so that the angle it subtends at a fixed point O inside the circle is bisected internally by the diameter through O, then PQ is perpendicular to that diameter. Let C be the centre of the circle and let CO, PQ intersect at N. Prove $\triangle OPN \equiv \triangle OQN$.

268. Let C be the centre of the circle, and let PQ, CO intersect at T.

[If CO bisects $\angle$ POQ, it must be the internal bisector of the angle.]

Through O draw a line at rt. $\angle$s to CO; PQ will be either (i) parallel to this line, or (ii) intersect it at R, say.

In case (ii) $O\{PQ, TR\} = -1$, $\therefore$ OR is the polar of T; and OR is fixed, $\therefore$ T is fixed.

$\therefore$ PQ is either parallel to OR or passes through a fixed pt.

269. With the construction given, we get $X\{DB, YZ\} = -1$,
$\therefore \{CA, YT\} = -1$, $\therefore$ T and Z are both on the polar of Y,
$\therefore$ the polar of Y passes through X.

271. Let PQ cut the circle at R and S.
$\because$ the polar of P passes through Q, $\therefore \{RS, PQ\} = -1$.
$\therefore$ the polar of Q passes through P.

272. Let P move on the fixed straight line XY; and let Q be the pole of XY with respect to the circle.
Since the polar of Q (viz. XY) passes through P, $\therefore$ the polar of P passes through Q, i.e. passes through a fixed point.

273. Let the straight line AB always pass through the fixed point C.
Since AB passes through C, $\therefore$ the polar of C passes through the pole of AB, i.e. the pole of AB lies on the fixed line, the pole of the fixed point C.

274. Let P be the point at which the polars of A and B intersect.
Since P is on the polar of A, $\therefore$ A is on the polar of P.
Again P is on the polar of B, $\therefore$ B is on the polar of P.
$\therefore$ AB is the polar of P.

275. This is a particular case of Ex. 274.
BC is the polar of A and PQ is the polar of P.
$\therefore$ AP is the polar of Q.

276. By Ex. 275, AP is the polar of Q, $\therefore$ if AP cuts BC at R,
$\{BC, RQ\} = -1$, $\therefore$ A $\{BC, PQ\} = -1$.

277. X′ is on the polar of A, $\therefore$ A is on the polar of X′. Also X is on the polar of X′, $\therefore$ AX is the polar of X′,
$\therefore$ A $\{BC, XX'\} = -1$, $\therefore \{BC, XX'\} = -1$.

278. Let ZW intersect PT at R and XY at S.
$\because$ P $\{ZW, YT\} = -1$, $\therefore \{ZW, SR\} = -1$,
$\therefore$ the polar of R passes through S. Also the polar of P passes through R, $\therefore$ R is the pole of PS, i.e. of XY,
$\therefore$ R is a fixed point, $\therefore$ PT passes through a fixed point.

279. Let XY, PQ intersect at W.

$\therefore$ P $\{XY, ZQ\} = -1$, $\therefore$ $\{XY, ZW\} = -1$, $\therefore$ Q $\{XY, ZW\} = -1$.

Now P, Q correspond to two points on the fixed line XY of Ex. 278. $\therefore$ PZ, QZ correspond to two positions of the line PT.

$\therefore$ by Ex. 278, Z is a fixed point, the polar of PQ.

$\therefore$ XY passes through the polar of PQ.

280. (i) Any line drawn from a point T to cut the polar of T with respect to a circle of infinite radius is bisected by the circumference of the circle.

Or: If AR and NH are two parallel straight lines and T a point on AN such that TA = AN, then any line drawn from T to cut AR, NH at R, H is such that TR = RH.

(ii) If a point T is joined to any point R on a fixed straight line and TR is produced to H so that RH = TR, the locus of H is a fixed straight line.

(iii) The fixed point would be at infinity, so that we should only arrive at the fact that all the polars are parallel (which we already know as they are all parallel to the circumference of the circle).

(iv) Nothing easily intelligible comes from this particular case.

281. Let BC be the polar of A.

Then B $\{PP', AC\} = -1$, and $\angle$ BAC is a rt. $\angle$.

$\therefore$ BA bisects $\angle$ PBP'.

Similarly BA bisects $\angle$ QBQ'.

$\therefore$ PQ, P'Q' subtend equal angles at B.

282. Let X be on PQ produced; draw XY a tangent to the circle cutting TP, TQ at Z, W.

First prove that YT is the polar of X; then by Theorem 33 $T_1\{PQ, YX\} = -1$, $\therefore$ $\{ZW, YX\} = -1$.

283. Let the tangent at R cut PQ at S, and let TR, PQ intersect at X.

S is on the polar of T, $\therefore$ T is on the polar of S; also R is on the polar of S, $\therefore$ TR is the polar of S.

$\therefore \{PQ, XS\} = -1$, $\therefore$ R $\{PQ, XS\} = -1$.

$\therefore$ HK is bisected at T ($\because$ HK is $\parallel$ to RS).

Or thus: Let the tangent at R cut TP, TQ at X, Y.

Then by similar $\triangle$s

$$TH : XR = TP : XP \text{ but } XR = XP, \therefore TH = TP.$$

Similarly $TK = TQ$ and $TP = TQ$, $\therefore$ $TH = TK$.

284. Let C be the centre of the given circle.

Then $$OQ \cdot ON = ON^2 - NQ \cdot NO = ON^2 - NP^2 = CO^2 - CP^2$$
$$= CO^2 - CT^2 = OT^2 = OT'^2.$$

$\therefore$ OT, OT′ touch the circles TQN, T′QN.

285. Let AR cut the circle at X, Y.

Then $QP = QA = QR$. $\therefore$ $QX \cdot QY = QP^2 = QR^2$.

$\therefore \{XY, AR\} = -1$, $\therefore$ R lies on the polar of A.

286. Let OC cut the circle in A, B; let TS cut NO in X.

Having regard to sense, $$XO^2 = XS \cdot XT = XA \cdot XB$$
$$= CX^2 - CA^2 = CX^2 - CN \cdot CO = CX^2 - (CX + XN)(CX + XO)$$
$$= -CX \cdot XN - CX \cdot XO - XN \cdot XO.$$

$\therefore (XO + CX)(XO + XN) = 0$. Now $XO + CX \neq 0$,

$\therefore XO + XN = 0$, $\therefore$ X is the mid-point of ON.

Or thus: Let OS cut the circle ABT in Y.

Then $\angle XOS = \angle STO = \angle SYT$. $\therefore$ TY is $\parallel$ to OX, and T, T′, S, Y subtend a harmonic pencil at T (Ex. 265).

$\therefore$ T $\{ON, SY\} = -1$, $\therefore$ TS bisects ON.

287. Let the tangents at A, B intersect at R, and those at C, D at S. Then RS is the polar of O (Ex. 260).

$\therefore$ R $\{PQ, SO\} = -1$, and S $\{PQ, RO\} = -1$ (Theorem 33).

$\therefore$ O, P, Q are collinear.

[The last step is a particular case of Theorem 56; it can easily be proved as follows:—

Let OP cut QR, QS, RS in q, q', T.
Now R $\{PQ, SO\} = -1$, $\therefore$ $\{Pq, TO\} = -1$.
And S $\{PQ, RO\} = -1$, $\therefore$ $\{Pq', TO\} = -1$.
$\therefore$ q, q' coincide, $\therefore$ Q is on OP.]

288. Let TP, TQ be the tangents, R any point on the circumference; draw RX, RY, RZ perpendicular to TP, TQ, PQ respectively.
Now RX = RP sin XPR, RY = RQ sin YQR,
and RZ = RP sin ZPR = RQ sin ZQR.
But $\angle$ XPR = $\angle$ ZQR and $\angle$ YQR = $\angle$ ZPR, $\therefore$ RX . RY = RZ^2.

289. Let X, Y, Z be the points at which the incircle touches the sides.
YZ is the polar of A; and the polar of D is the perpendicular from P to ID or YZ. Let this perpendicular cut YZ at α. Then α is the pole of AD.
Similarly β, γ, the feet of the perpendiculars from P on ZX, XY are the poles of BE, CF.
Now α, β, γ lie on the Simson line of P with respect to the incircle.
Since α, β, γ are collinear, their polars, AD, BE, CF, are concurrent (Ex. 272).

290. Let P, Q be the two points, C the centre of the circle; and let CP, CQ cut the polars of P, Q at N, M. Draw PX, QY $\perp$ to the polars of Q, P; and draw PR, QS $\perp$ to CQ, CP.
By similar $\triangle$s, CP : CQ = CR : CS.
$\because$ CP . CN = (radius)^2 = CQ . CM,
$\therefore$ CP : CQ = CM : CN.
$\therefore$ CP : CQ = CM $-$ CR : CN $-$ CS = PX : QY.

291. Let T be on the polar of O. Draw TP, TQ tangents to the circle. As T is on the polar of O, O is on the polar of T.
$\therefore$ POQ is a straight line.
Let PQ cut the polar of O in R. Draw OX, OY $\perp$ to TP, TQ, and OL, PM, QN $\perp$ to TR. Then RT is the polar of O, and PT the polar of P.

$\therefore$ CO : CP = OX : PM. Similarly CO : CQ = OY : QN.

Again RT is the polar of O. $\therefore \{PQ, RO\} = -1$.

$$\therefore \frac{1}{RP} + \frac{1}{RQ} = \frac{2}{RO}.$$

$$\therefore \frac{1}{PM} + \frac{1}{QN} = \frac{2}{OL}.$$

$$\therefore \frac{CO}{CP} \cdot \frac{1}{OX} + \frac{CO}{CQ} \cdot \frac{1}{OY} = \frac{2}{OL}.$$

$$\therefore \frac{1}{OX} + \frac{1}{OY} = \frac{2}{OL} \cdot \frac{CP}{CO} = \text{const.}$$

$\therefore$ the harmonic mean between OX, OY is constant.

CHAPTER VIII.

293. Let C be the centre of the given circle and P any point on it.

Let C′, P′ be the mid-points of OC, OP.

Then C′P′ : CP = OC′ : OC = 1 : 2.

$\therefore$ C′P′ = $\frac{1}{2}$ CP.

$\therefore$ the locus of P′ is a circle with centre C′.

294. $\frac{OP}{OQ} = \sin POQ$, $\therefore$ OQ is constant, $\therefore$ locus of Q is a circle.

295. Draw a line through P $\perp$ to PA, and cut off Pa = PA, etc.

Let AC, ac intersect at X.

First prove $\triangle P ca \equiv \triangle PCA$.

$\therefore$ $\angle Pac = \angle PAC$.

Hence, in quadrilateral PAXa, since $\angle APa = 90°$,

$$\angle AXa = 90°.$$

296. A circle whose centre C′ is the intersection of CC′ drawn at rt. $\angle$s to OC and OC′ drawn so that $\angle COC' = \angle POQ$ (in magnitude and sense).

This could be proved independently from the similarity of $\triangle$s COP, C′OQ.

299. Draw a line through the centre of similitude O making a small angle with OPP′ and cutting the curves at Q, Q′.
Then OP : OP′ = OQ : OQ′, ∴ PQ is ∥ to P′Q′.
Now move OQ up to the position OP; in the limit, we have the tangents at P and P′ parallel.

300. Suppose OP touches one of the curves at P, then we may regard OP as cutting the curve in two coincident points at P. ∴ OP cuts the other curve at two coincident points at the point corresponding to P. ∴ OP touches the other curve and the points of contact are corresponding points.

301. Let D, E, F be the centres of the circles A, B, C. Let PQ cut EF at O and the circle B at R.
Then ∠ DQP = ∠ DPQ = ∠ EPR = ∠ ERP.
∴ ER is ∥ to QF.
∴ OE : OF = ER : FQ.
∴ O is a centre of similitude of the two circles.

302. Let C, C′ be the centres of the two circles; let PP′ touch them at P, P′ and cut EF at O.
Then △s OCP, OC′P′ are similar.
∴ OC : OC′ = CP : C′P′.
∴ O is a centre of similitude of the two circles.

303. The two ends of the diameter perpendicular to the given line are the two centres of similitude.
[Note that the ratio is zero or infinite.]

304. Any point whatever is a centre of similitude.

305. The line of centres contains two distinct points at infinity, and is therefore the line at infinity. The centres of similitude are therefore at infinity.

306. ∠ BAC is constant and AB : AC is constant, ∴ by § 3, p. 72, B and C describe similar loci.

307. See fig. 23. The centres of similitude are the two points which divide SN internally and externally in the ratio 2 : 1; these are G and H, the centroid and the orthocentre.

308. (i) Let OA be the perpendicular from O to the line, and B the point corresponding to A.

Then $\angle AOB = \angle XOY$, $\therefore \angle AOX = \angle BOY$ and $OA : OB = OX : OY$, $\therefore \angle OAX = \angle OBX$, $\therefore$ $\angle OBY$ is a rt. $\angle$, $\therefore$ the locus of Y is a straight line through B at right angles to OB.

Or thus: $OX : OY$ = const. and $\angle XOY$ is const.
$\therefore$ the locus of Y is similar to that of X by § 3.
$\therefore$ the locus of Y is a straight line.

(ii) Let C be the centre of the circle, and C′ the point corresponding to C.

Then $\angle COC' = \angle XOY$, $\therefore \angle COX = \angle C'OY$, also $OC : OC' = OX : OY$, $\therefore$ $CX : C'Y = OC : OC'$ = const.
$\therefore$ C′Y is const. $\therefore$ the locus of Y is a circle.

Or thus: As in (i) the locus of Y is similar to the locus of X; and is, therefore, a circle.

309. Take the $\triangle ABC$; call the inscribed circle P, and Q the escribed circle opposite A. Draw the tangent to P at the point furthest from BC and let it cut AB, AC at B′, C′. Then B′C′ is $\parallel$ to BC.

Now the figures P, B′C′ and Q, BC have A as centre of similitude, $\therefore$ the point of contact of B′C′ with P and that of BC with Q are collinear with A.

310. [Misprint in Ex. 310; for P read D.]

Let AD cut the circle at E. Then $AD \cdot DE = BD \cdot DC$; but, by hypothesis, $AD^2 : BD \cdot DC$ = const., $\therefore$ $AD : DE$ = const.
$\therefore$ $AD : AD + DE$ = const., i.e. $AD : AE$ = const.
$\therefore$ the locus of D is similar to that of E, A being the centre of similitude.
$\therefore$ the locus of D is a circle touching the given circle at A.

311. Let AC, OD intersect at P.

Consider $\triangle ABC$, $\frac{AO}{BO} \cdot \frac{BD}{CD} \cdot \frac{CP}{AP} = 1$, $\therefore$ $AP = 2PC = \frac{2}{3} AC$.

$\therefore$ the locus of P is a circle, touching the given circle at A and having its diameter $= \frac{2}{3} AB$.

312. The triangle ABC is fixed; and CD is given in length, $\therefore$ the locus of D is a circle with centre C. $\therefore$ the locus of P, the mid-point of BD, is similar to the locus of D, and is therefore a circle.

Again Q, the mid-point of AC, is fixed. $\therefore$ the locus of the mid-point of PQ is similar to the locus of P and is therefore a circle.

313. Let C be the centre of the circle. From CO cut off $CC' = \frac{1}{5} CO$; with centre C′ and radius equal to $\frac{4}{5}$ of the radius of the given circle, draw a circle to cut the given circle at P; and let OP cut the given circle again in Q.

O is the centre of similitude of the two circles,

$$\therefore\ OP = \tfrac{4}{5} OQ. \quad \therefore\ PQ = \tfrac{1}{5} OQ.$$

[Another solution may be obtained for the case in which Q lies between O and P. In this case $CC' = \frac{1}{6} CO$, and the radius of the new circle $= \frac{5}{6}$ of the radius of the given circle.]

314. [By putting C at B, D, and the ends of the diameter perpendicular to BD, we get several positions of P, from which we see that the locus may consist of two straight lines parallel to BD and at a distance equal to half a radius from it.]

Let OP, AC intersect at Q; draw PM, QN $\perp$ to BD.

$\triangle APO \equiv \triangle APQ$, $\therefore$ $\triangle OCQ \equiv \triangle QNO$, $\therefore$ $QN = OC$.

But $PM = \frac{1}{2} QN = \frac{1}{2} OC = \text{const.}$

$\therefore$ P is at a constant distance from BD.

$\therefore$ the locus of P is a pair of straight lines parallel to BD at a distance $= \frac{1}{2} OC$ from BD.

315. Let ABC be the given equilateral $\triangle$, and PQ the given straight line. On PQ describe an equilateral $\triangle$ PQR. Let S, T be the circumcentres of $\triangle$s ABC, PQR.

With S as centre and radius = TP, describe a circle cutting each side of $\triangle$ABC in two points; let the points in AB, BC, CA nearer to A, B, C respectively be X, Y, Z.

Then $\triangle$s AXS, BYS, CZS can be proved congruent.

$\therefore$ by § 3, $\triangle$XYZ is similar to $\triangle$ABC, and is therefore equilateral; and XY : AB = SX : SA = TP : SA = PQ : AB.

$\therefore$ $\triangle$XYZ is the required equilateral $\triangle$ with its sides equal to PQ.

[The other points in which the circle XYZ cuts the sides of $\triangle$ABC would give another solution.]

316. Let PQR be any $\triangle$ of the given species; let A be the given point and OX, OY the given lines.

Let B be a moving point on OX, and let $\triangle$ABC be similar to $\triangle$PQR.

Then $\angle$ BAC = $\angle$ QPR and AB : AC = PQ : PR.

$\therefore$ the locus of C is similar to the locus of B.

$\therefore$ the locus of C is a straight line.

Then the required position of C is where this line cuts OY, etc., etc.

317. OP . OQ = const. = t^2 say. OP . OR = k^2. $\therefore$ OQ : OR = $t^2 : k^2$.

$\therefore$ the locus of R is similar to that of Q and is therefore a circle.

CHAPTER IX.

318. Let A, B be the centres of the circles.

AP, BP are at rt. $\angle$s to the tangents at P.

$\therefore$ $\angle$ APB = one of the $\angle$s between the tangents at P.

Similarly $\angle$ AQB = one of the $\angle$s between the tangents at Q.

But $\angle$ APB = $\angle$ AQB. $\therefore$ etc.

319. Let A, B be the centres of the circles, P one of their points of intersection.

Draw PT the tangent at P to the 'A' circle, $\therefore \angle$ APT is a rt. $\angle$. But the circles cut orthogonally, $\therefore$ as PT is a tangent to the 'A' circle, PA is a tangent to the 'B' circle. And $\therefore$ PT, which is at rt. $\angle$s to the tangent PA, passes through B.

320. Let O be the centre of $\odot$ B, and OT a tangent to $\odot$ A.
OT = the radius of B, $\therefore$ T is on the circumference of B.
$\therefore$ the tangent to B at T is at rt. $\angle$s to OT.
$\therefore$ the circles are orthogonal.

321. Let P, Q be the given points, and let the tangents at P, Q intersect at C. With centre C and radius CP describe a circle; this is the required circle.

If P, Q are opposite ends of a diameter, the point C will be at infinity and the required circle will consist of the diameter PQ together with the line at infinity.

322. Let P be the given point. Along the tangent at P, cut off PC equal to the given radius. With centre C and radius CP describe a circle; this is the required circle.

Yes.

323. With centre P and radius equal to one of the tangents from P describe a circle; this circle will cut the given circle orthogonally. Proof as in Ex. 320.

324. Let P, Q be the common points of the two orthogonal circles A and B. The tangents to A at P and Q pass through the centre of B by Theorem 34; but the point of intersection of these tangents is the pole of PQ.

325. Draw the tangents at one of the points of intersection, these pass through the centres of the circles. Apply Pythagoras' Theorem.

326. *If the sum of the squares on the radii of two circles is equal to the square on the distance between their centres, the circles cut orthogonally.*

Let A, B be the centres of the circles, and P a point of intersection. $\angle APB$ is a rt. $\angle$ (converse of Pythagoras' Theorem). $\therefore$ PA, PB are tangents and are at rt. $\angle$s.

327. Let AB be divided harmonically at C, D; let O be the mid-point of AB. The square on tangent from O to $\odot$ on CD as diameter $= OC \cdot OD = OB^2$, $\therefore$ the circles cut orthogonally (see Ex. 320).

328. Let O be the centre and AB a diameter of one of the circles, let AB cut the other circle in C, D.

$OC \cdot OD$ = the square on the tangent from $O = OB^2$.

$\therefore \{AB, CD\} = -1$.

329. Let O be the centre of the given circle.

The square on the tangent from O to any circle through P, Q $- OP \cdot OQ$ = the square on the radius of the given circle. $\therefore$ the circles cut orthogonally.

330. Let O be the centre of the given circle, and P the fixed pt. Let OP cut one of the variable circles again at Q.

$\because$ the circles cut orthogonally, $\therefore$ the tangent from O to the variable $\odot$ – the radius of the given $\odot$, $\therefore$ $OP \cdot OQ =$ (radius of given $\odot)^2$, $\therefore$ Q is a fixed point.

331. Let P and R be the given points, and O the centre of the given circle.

In OP find Q such that $OP \cdot OQ =$ (radius of given $\odot)^2$.

Then any orthogonal circle through P passes through Q (Ex. 330).

Through P, Q, R describe a circle; this is the required circle.

If O, P, R lie in a straight line, P, Q, R lie in a straight line. Therefore in this case the circle degenerates into the straight line OPR with the line at infinity.

If Q coincides with R, an infinite number of circles can be described to satisfy the conditions.

332. Draw PT, PV tangents at P to the two circles; let T and V be on the side of APB opposite to Q.

$\angle APT = \angle AQP$, and $\angle BPV = \angle BQP$; but the circles cut orthogonally, $\therefore \angle TPV =$ a rt. $\angle$, $\therefore \angle AQB =$ a rt. $\angle$.

333. Let P, Q be the common points of the two $\odot$s; and let $\angle$s PAQ, PBQ be the complementary angles.

Draw PX, PY to touch the circles PBQ, PAQ.

Then $\angle XPQ = \angle PBQ$, and $\angle YPQ = \angle PAQ$.

$\therefore \angle XPY$ is a rt. $\angle$.

$\therefore$ the circles cut orthogonally.

334. Let P, Q be the fixed points on the $\odot$, AB the variable diameter and let AP, BQ intersect at R.

(i) When R is outside the circle.

$\angle PRQ$ is complementary to the angle in the major segment on PQ (since AB is a diameter, and the angle in a semicircle is a rt. $\angle$).

$\therefore$ the locus of R is a circle through P, Q orthogonal to the given circle.

(ii) When R is inside the circle.

Let BP, AQ intersect at R′. Then R′ is outside the given circle and on the orthogonal circle found above.

$\because$ AB is a diameter, $\therefore$ $\angle$s APR′, BQR′ are rt. $\angle$s.

$\therefore$ P, Q, R, R′ are concyclic.

$\therefore$ the locus of R in this case also is the circle found above.

335. Let QR be the given base, and let P be the required vertex.

Since PQ : PR is given, $\therefore$ the locus of P is an Apollonius circle.

Again since the area of the triangle is given and the base, the locus of P is a pair of straight lines $\parallel$ to QR.

The intersections of these loci give the positions of P.

There seem to be four solutions in general, but it can be seen that there are really two pairs of congruent triangles so that in general there are only two solutions.

336. As in Ex. 335, P lies on an Apollonius circle and also on a segment of a circle on QR as base.
There is one solution.

337. Let A, B, C be the given points and P the required point.
Since PA : PB = const., ∴ the locus of P is a circle.
Since PB : PC = const., ∴ the locus of P is a circle.
The points of intersection of these two circles are the positions of P.
In general there are two solutions.

338. Let ABC be the required triangle, and D the mid-point of BC. Let the bisector of ∠ A cut BC at E. Then D, E are given and the direction of EA.
Now BE : EC = AB : AC (given).
Take any line $\beta\gamma$, bisect it at δ, and divide it at ϵ so that $\beta\epsilon : \epsilon\gamma = AB : AC$. In DE produced, find a point C such that $DE : EC = \delta\epsilon : \epsilon\gamma$; from CD produced cut off DB = CD.
Since AB : AC is given, the locus of A is a circle through ϵ; draw this circle; it cuts the line of the bisector again at A.

339. Since $\{QR, XY\} = -1$, ∴ ⊙s PQR, PXY cut orthogonally, ∴ the tangent at P to circle PXY passes through the centre of ⊙ PQR (Theorem 34).

340. Let ABC be the triangle. The circle obtained from the bisectors of ∠ A is the locus of points which move so that the ratio of their distances from B, C = AB : AC. Similarly for the other two circles. Let these two latter circles intersect at P, Q. Then PC : PA = BC : BA and PA : PB = CA : CB. ∴ PB : PC = AB : AC. ∴ P lies on the circle obtained from the bisectors of ∠ A. Similarly for Q. ∴ the three circles pass through P and Q.

341. Suppose R and S coincide. The theorem reduces to the identity PR . RQ = PR . RQ.

342. If P, Q, R, S are points on a straight line
PS . RQ + PQ . SR = PR . SQ. See Ex. 2.

343. By Ptolemy's Theorem $PA \cdot BC = PB \cdot CA + PC \cdot AB$; but $BC = CA = AB$. $\therefore PA = PB + PC$.

344. Let P be the point whose locus is required.
Then $PA = PB + PC$, and $BC = CA = AB$.
$\therefore PA \cdot BC = PB \cdot CA + PC \cdot AB$.
$\therefore$ ABPC is a cyclic quadrilateral.
$\therefore$ the locus of P is the minor arc BC of the circle ABC.

345. Take the cyclic quadrilateral PQRS (take the letters in order), PQ is to be a diameter and $\angle QPR = \alpha$, $\angle PQS = \beta$.
$\therefore \angle SPQ = 90 - \beta$. $\therefore \angle SPR = 90 - (\alpha + \beta)$.
By Ptolemy's Theorem $SR \cdot PQ + PS \cdot RQ = PR \cdot SQ$.
And $SR = 2\rho \sin(90 - \overline{\alpha + \beta}) = 2\rho \cos(\alpha + \beta)$,
$PQ = 2\rho$, $PS = 2\rho \sin\beta$, $RQ = 2\rho \sin\alpha$, $PR = 2\rho \cos\alpha$,
$SQ = 2\rho \cos\beta$, $\therefore$ etc.

346. (i) Take the cyclic quadrilateral PQRS with PQ a diameter.
Let $\angle SPQ = \alpha$ and $\angle RPQ = \beta$, $\therefore \angle SPR = \alpha - \beta$, etc., etc. as in Ex. 345.

(ii) Take the cyclic quadrilateral PQRS with PR a diameter.
Let $\angle QPR = \alpha$, and $\angle PRS = \beta$, $\therefore \angle SPR = 90 - \beta$,
$\therefore \angle QPS = 90 + \overline{\alpha - \beta}$,
$\therefore SQ = 2\rho \sin(90 + \overline{\alpha - \beta}) = 2\rho \cos(\alpha - \beta)$, $\therefore$ etc.

347. P, L, or C may have the suffix 3 (the other suffixes being 0).
P, L, or C may have the suffix 2, in each of which cases there are two ways of choosing which of the remaining letters has the suffix 1.
Or P, L, C may each have the suffix 1.
$\therefore$ the number of cases $= 3 + 6 + 1 = 10$.

348. $P_3L_0C_0$. One case only. [*Elementary Geometry*, p. 224.]
$P_0L_3C_0$. Four cases. [*Elementary Geometry*, pp. 243, 244.]

349. Let A, B be the given points. Let AB produced cut the given line at T. Draw any circle through A, B; and from T draw TD the tangent to it; on the given line make $TC = TD$. Describe the circle ABC.

$\therefore TC^2 = TD^2 = TA \cdot TB$, $\therefore$ TC touches this circle at C.

In general there are two solutions, for TC may be measured along the given line in either direction.

350. Let A, B be the given points. Draw any circle through A, B to cut the given circle at P, Q. Let AB, PQ cut at T. From T draw TC to touch the given circle at C. Describe a circle through A, B, C. This is the required circle, for it touches TC at C ($TA \cdot TB = TP \cdot TQ = TC^2$), and therefore touches the given circle at C.

In general there are two solutions, for two tangents may be drawn from T to the given circle.

351. Let A be the given point, and OX, OY the given lines. Draw any circle to touch OX, OY on the same side of OX, OY as A. Let OA cut this circle in A′, and let OX, OY touch the circle at B′, C′; through A draw AB $\parallel$ to A′B′ to cut OX at B, and AC $\parallel$ to A′C′ to cut OY at C. Describe the circle through ABC. This is the required circle; for the figure ABC is similar to the figure A′B′C′, O being the centre of similitude.

In general there are two solutions, since OA cuts the construction circle in two points.

352. Let X be the centre of the circle which touches the line CN at N and the circle with centre Y at M; let MN cut the 'Y' circle again at A.

Now $\angle YAM = \angle YMA$ and $\angle XNA = \angle XMN$; but $\angle$s YMA, XMN are either equal or the same angle.

$\therefore \angle YAM = \angle XNA$, $\therefore$ YA is $\parallel$ to XN, $\therefore$ YA is $\perp$ to CN.

353. Follow the directions given; then describe a circle through P and P′ to touch the given line. See Ex. 349.

In general there are four solutions, since P may be joined to either end of the diameter perpendicular to the given line, and two circles can be described to pass through two points and touch a given line.

354. Let M, N be the points of contact of the required circle with the given circle. Then MN passes through one of the centres of similitude of the two circles (Ex. 301).

(i) Suppose that MN passes through O, their external centre of similitude.

Draw the straight line OC′BCB′ through the centres of the given circles. Let OMN cut the circle C′BM again at N′.

Then $\angle ONC = \angle ON'C'$ (O is the centre of similitude of the two circles), and $\angle ON'C' = \angle OBM$ ($\because$ B, M, N′, C′ are concyclic), $\therefore \angle ONC = \angle OBM$.

$\therefore$ M, N, B, C are concyclic, $\therefore$ OM . ON = OB . OC.

Let P be the given point; in OP take P′ such that OP . OP′ = OB . OC. Then describe a circle to pass through P, P′ and touch one of the given circles.

By reversing the above steps it can be shown that this circle will also touch the other given circle.

(ii) The above will need slight modification for the case in which MN passes through the internal centre of similitude.

In general there are four solutions; for two circles can be described to pass through two points and touch a given circle so that each case admits of two solutions.

355. See directions given in text.

In general there are eight solutions; for two circles can be described to pass through a given point and touch two given lines, and the parallels may be drawn on either side of the given lines. The eight solutions are all real if the given lines intersect inside the given circle.

356. Call the given circles A, B; let a, b be their radii ($a < b$).

Consider any circle D which touches A and B and the given line, L.

A circle can be drawn concentric with D which (i) passes

through the centre of A, (ii) touches a circle concentric with B of radius $b \pm a$, (iii) touches a line parallel to L at a distance a from it.

Hence we may say that, by parallel translation, the given problem reduces itself to the subsidiary problem of describing a circle having given $P_1L_1C_1$.

In general there are eight solutions; for in making the parallel translation there are two possible positions for the line and two for the circle concentric with B; this seems to indicate sixteen solutions (the subsidiary problem has four solutions), but a figure will show that half the solutions of the subsidiary problem do not lead to solutions of the given problem.

Or we may see that there are only eight solutions by regarding the circle A as decreasing in radius and becoming a point-circle (the circle B and the given line remaining fixed); every solution of $P_1L_1C_1$ represents the coincidence of two solutions of $P_0L_1C_2$, for we can find pairs of circles which touch B on opposite sides which will coincide when B becomes a point-circle; and as there are four solutions of $P_1L_1C_1$, there are, in general, eight solutions of $P_0L_1C_2$.

357. Reduce one of the circles to a point by the method of parallel translation; the problem is thus reduced to describing a circle having given $P_1L_0C_2$.

In general there are eight solutions. As in the last Ex., there seem to be sixteen solutions of the subsidiary problem, but in this case also only half of them lead to solutions of the given problem.

Or we may see that there are only eight solutions by reasoning similar to that of Ex. 356; for each solution of $P_1L_0C_2$ represents the coincidence of two solutions of $P_0L_0C_3$.

358. Let A be the given point, B the centre of the given circle and r its radius.

Let P be the centre of any circle which passes through A and cuts the given circle orthogonally. Draw PN $\perp$ to AB.

$BP^2 = r^2 + PA^2$ (Theorem 35), $\therefore$ $BN^2 - NA^2 = r^2$, $\therefore$ N is a fixed point.

$\therefore$ the locus of P is a straight line through N $\perp$ to AB.

359. Let P, Q be the centres of the two circles. Let AP cut the orthogonal circle at N.

Then PN . PA = (tangent from P to $\odot$ ABN)^2 = (radius of 'P' circle)^2; and $\angle$ ANB is a rt. $\angle$, $\therefore$ NB is the polar of A.

Similarly the polar of A with respect to the 'Q' circle passes through B.

360. Let O be the common point nearest to BB'.

Let P, P' be the centres of the two circles.

$$\angle AOA' = \angle PAO + \angle P'A'O = \angle POA + \angle P'OA'.$$

But $\angle AOA' + \angle POA + \angle P'OA' =$ a rt. $\angle$.

$$\therefore \angle AOA' \quad \tfrac{1}{2} \text{ a rt. } \angle.$$

Similarly $\angle BOB' = \angle PBO + \angle P'B'O = \angle POB + \angle P'OB'$.

And $\angle BOB' + \angle POB + \angle P'OB' = 3$ rt. $\angle$s,

$$\therefore \angle BOB' - \tfrac{3}{2} \text{ of a rt. } \angle,$$

$$\therefore \angle AOA' + \angle BOB' = 2 \text{ rt. } \angle \text{s}.$$

361. Let BX, AY intersect at Q.

First consider the case in which BX, AY intersect without being produced.

$$\angle AQB = \angle AXB + \angle XAQ$$
$$= \angle AXB + \angle APB + \angle AYB$$
$$= \text{const. } (\because \text{ the arcs AB are fixed}),$$

$\therefore$ the locus of Q is an arc of a circle.

The above proof will need modification for the other cases.

362. Let A, B be the centres of the two circles; let P be a point on the locus.

Draw PQ, PR tangents to the two circles.

$\angle$s APQ, BPR are half the $\angle$s subtended at P by the circles.

$\therefore$ $\triangle$s APQ, BPR are similar. $\therefore$ AP : BP = AQ : BR = const.
$\therefore$ the locus of P is a circle (Theorem 37).

363. By Theorem 37 the locus of P is a circle.
Let the locus cut AB internally in X and externally in Y.
Since $m > 1$, Y is in $\overline{AB}$ produced.

$$AX = m \cdot XB, \quad \therefore AB = AX + XB = (m+1)\,XB.$$
$$AY = -m \cdot YB, \quad \therefore AB = AY - BY = (m-1)\,BY.$$
$$\therefore XY = AB\left(\frac{1}{m+1} + \frac{1}{m-1}\right) = AB\,\frac{2m}{m^2-1}.$$
$$\therefore \text{ the radius of the circle} = AB\,\frac{m}{m^2-1}.$$

Since AXBY is a harmonic range,
$\therefore$ the polar of A passes through B.

364. CA : CA′ = AB : A′B′ = const. $\therefore$ C lies on an Apollonius circle, A, A′ being the fixed points.
Again $\angle$ BAC = $\angle$ B′A′C, in the same sense; $\therefore$ the locus of C is a circle through A, A′.
This circle cuts the Apollonius circle in two real points, for the Apollonius circle cuts AA′ internally and externally.
$\therefore$ there are always two positions of C.

365. Let the chords AA′, BB′, CC′ intersect at X.
By similar $\triangle$s $\dfrac{AB'}{BA'} = \dfrac{XA}{XB}$, $\dfrac{BC'}{CB'} = \dfrac{XB}{XC}$, $\dfrac{CA'}{AC'} = \dfrac{XC}{XA}$,
$\therefore$ AB′ . BC′ . CA′ = BA′ . CB′ . AC′.

366. Let C, O be the centres of the orthogonal circles; and let a line cut the 'C' circle in P, P′ and the 'O' circle in Q, Q′; suppose {PP′, QQ′} = −1. Draw CN $\perp$ to PP′.
Then (by Theorem 26), NQ . NQ′ = NP² = CP² − CN².
But NQ . NQ′ = (tangent from N to 'O' circle)² = NO² − OQ².
$\therefore$ CP² − CN² = NO² − OQ².
But CP² + OQ² = CO² (the circles are orthogonal).
$\therefore$ CO² = CN² + NO².
$\therefore$ either C and N coincide, or $\angle$ CNO is a rt. $\angle$.
In the latter case PP′ passes through O.
$\therefore$ the chord must pass through C or O.

367. See Ex. 340.

368. Make a rough sketch of $\triangle ABC$, draw AD the bisector of $\angle A$, produce BA to X so that $AX = AC$, join XC.

Then AD is $\parallel$ to XC, and we are given AD, $\frac{BC}{BX}$, and $\angle B - \angle C$.

Now construct the required figure by the help of the rough sketch. Draw AD.

Now $BD : BA = BC : BX$, and $CD : CA = CD : XA = BC : BX$.

$\therefore$ D, C lie on an Apollonius circle; draw this circle.

Again $\angle ADC = B + \frac{1}{2}A = 90^\circ + \frac{1}{2}B - \frac{1}{2}C$; so make $\angle ADC$ and the line DC will cut the circle in B and C.

369. Let OX, OY be the given lines, and P one of the vertices of the rhombus. Let OP cut the circle in Q. Along OP cut off $\overline{OR} = \overline{PQ}$.

A circle equal to the given circle can be described through R to touch OX, OY; if this circle is now magnified in the ratio $OR : OQ$, the circle so obtained will pass through Q and, by the principle of similarity, it will touch OX, OY and the tangent to it at Q will be parallel to the tangent at R, and therefore parallel to the tangent at Q to the given circle. Therefore this circle will touch the given circle at Q.

In general OP will cut the given circle in two points and there are four possible positions of P; so that in general there are eight solutions.

[It is interesting to make rough sketches to show when some of the solutions are imaginary.]

370. (i) See Ex. 349.

(ii) Let A, B be the given points, and let AB cut the given line in T.

Through AB draw any circle (of diameter > the given chord), in this circle draw a chord CD = the given chord; describe a concentric circle to touch the chord; from T draw a tangent to the inner circle

cutting the outer at E, F. From the given line cut off TG, TH = TE, TF.

Now TG . TH = TE . TF = TA . TB, ∴ A, B, G, H are concyclic.

Also GH = EF = CD = the given length.

∴ ABGH is the required circle.

(iii) Let A, B be the points through which the circle is to pass, and C the other given point.

If AC cuts the circle again in D, (tangent from C)^2 = CD . CA; hence CD can be determined, and the required circle drawn through A, B, D.

371. Let S be the mid-point of PR; draw AT, BX ⊥ to PQR.

Now $AS^2 + SB^2 = AT^2 + TS^2 + SX^2 + BX^2$

$= AT^2 + XR^2 + TQ^2 + BX^2$

($\because$ TS = XR, and SX = TQ)

$= AQ^2 + BQ^2$

$= AB^2$.

∴ ∠ ASB is a rt. ∠.

372. Invert with regard to O, the theorem becomes:—

A′, B′, C′, D′ are points on a circle and A′C′ is ⊥ to B′D′, let A′B′, C′D′ subtend angles θ, ϕ at any point O on the circle; then either $\theta + \phi = \frac{\pi}{2}$, or $\theta + (\pi - \phi) = \frac{\pi}{2}$, or $\phi + (\pi - \theta) = \frac{\pi}{2}$. In other words; then the acute angles subtended by A′B′, C′D′ are complementary; which is easily proved. ∴ the given theorem is true.

373. Let M be the mid-point of EF. Draw OD ⊥ to EF to cut AB at D.

$\because$ EM = MF, ∴ O{EF, MD} = −1,

∴ O{AB, CD} = −1, ∴ D is a fixed point.

Again ∠ BOD = ∠ BEF = ∠ BAF.

∴ △s BOD, OAD are similar.

∴ DB : DO = DO : DA.

∴ DO^2 = DB . DA = const. ∴ the locus of O is a circle.

374. Let ⊙s FBD, ABC cut at O.

Then the feet of the perpendiculars from O on the sides of △ABC and on DE are collinear.

∴ O is on the circle CDE.

Again $\angle DOF = \angle ABC$, or $180° - \angle ABC$,

and $\angle DOE = 180° - \angle ACB$, or $\angle ACB$,

according as $DE/DF \gtrless 1$.

∴ ∠ DOF, ∠ DOE, and FD/DE are constant.

∴ the figure ODEF is given in specie.

∴ $\angle FBO = \angle FDO$ = const., and O is on the circle ABC.

∴ O is a fixed point.

375. Let A, B be the centres of the two circles; let P be any point on the circle on SS′ as diameter. From P draw PQ, PR tangents to the two circles.

Now S, S′ divide AB in the ratio AQ : BR,

∴ AP : BP = AQ : BR (Theorem 37).

∴ △s QPA, RPB are similar.

∴ $\angle QPA = \angle RPB$.

∴ the whole angles subtended at P by the two circles are equal.

376. Suppose ABCD to be the required quadrilateral in which we are given AC, BD and the angle between them and the angles ABC, ADC.

Draw AC, and on opposite sides of it construct arcs AB′C, AD′C to contain angles equal to the given angles.

Draw AA′, CC′ ∥ to the direction of BD and equal to BD; on A′C′ describe an arc equal to the arc AB′C, and let this arc cut the arc AD′C at D; draw DB ∥ to A′A to cut the arc AB′C at B. Then ABCD is the required quadrilateral.

For ∠ ABC, ∠ ADC are equal to the given angles, BD is drawn ∥ to its proper direction and is of the given length since the arc A′DC′ was moved by parallel translation through a distance equal to the length given for BD.

377. Let D be the mid-point of AB.

(i) When AB cuts the circle at P, Q.
Let CD cut the circle again at E.
Then $\{AB, PQ\} = -1$, and $AD = DB$.
$\therefore DA^2 = DP \cdot DQ = DE \cdot DC$.
$\therefore$ E is a fixed point.
$\therefore$ the centre of the circle lies on the $\perp$ bisector of CE.
i.e. the locus of the centre of the circle is a line $\perp$ to CD.

(ii) When AB does not cut the circle, then A must be outside the circle. Let O be the centre of the circle and let OA cut the polar of A at A′; since A is outside the circle, $\therefore$ A′ is inside, $\therefore$ DA′ cuts the circle.
Let A′D cut the polar of A′ at B′. Then AA′BB′ is a rectangle. $\therefore DA^2 = DA'^2 = DE \cdot DC$, etc. as in the first case.

378. Let ABC be the $\triangle$, and let AB, AC pass through the fixed points P, Q. Draw AR $\parallel$ to BC to cut the $\odot$ APQ at R.
Then $\angle QAR = \angle ACB$, $\therefore$ R is a fixed point.
Draw AD, RS $\perp$ to BC; then $RS = AD = \text{const.}$
$\therefore$ BC always touches the $\odot$ with centre R and radius $= AD$.

379. Let ABC be the $\triangle$ and let AB, AC touch circles with centres P, Q. Through P, Q draw A′B′, A′C′ $\parallel$ to AB, AC to cut BC at B′, C′.
Then $\triangle$ A′B′C′ is fixed in size and shape, and its sides A′B′, A′C′ pass through P, Q.
$\therefore$ the envelope of B′C′ (i.e. of BC) is a circle.

CHAPTER X.

380. Let T be any point on PQ produced; draw TR, TS tangents to the two circles.

Then $TR^2 = TP \cdot TQ = TS^2$. $\therefore$ TR = TS.

381. Let A, B be the centres of the circles; and let the line through P cut the 'A' circle in W, X, and the 'B' circle in Y, Z. Draw PN $\perp$ to AB.

Then $PW \cdot PX = AP^2 - AX^2 = PN^2 + AN^2 - AX^2$,

and $PY \cdot PZ = BP^2 - BY^2 = PN^2 + BN^2 - BY^2$.

But P is on the rad. axis, $\therefore$ $AN^2 - BN^2 = AX^2 - BY^2$,

$\therefore$ $PW \cdot PX = PY \cdot PZ$.

[The above proof holds even in the case when P is on the common chord of two intersecting circles; in that case $AP^2 - AX^2$ is negative.]

382. Using the notation of Theorem 40, let $AN = x$.

Then $$x^2 - (d-x)^2 = R^2 - r^2,$$

$$\therefore x = \frac{R^2 - r^2 + d^2}{2d},$$

$$\therefore \frac{AN}{NB} = \frac{x}{d-x} = \frac{d^2 + R^2 - r^2}{d^2 - R^2 + r^2}.$$

$\therefore$ the values of $\frac{AN}{NB}$ are (i) 1·38, (ii) 1·67, (iii) 2·60, (iv) $-1\cdot67$, (v) $-1\cdot13$, (vi) -1, (vii) 3·08, (viii) $-2\cdot12$, (ix) -1, (x) 1.

383. In both cases the radical axis is the straight line which with the line at infinity constitutes the circle of infinite radius.

384. Their common tangent at their point of contact.

385. Let the common tangent touch the circles at Q, R and cut the rad. axis at P. By def. PQ = PR.

386. Draw two common tangents and join their mid-points, this line is the rad. axis of the circles.

This construction fails when one circle is inside the other.

387. Use the notation of Chap. IV., see fig. 17.
The rad. axis bisects XX_1, and $BX_1 = XC$,
$\therefore$ the rad. axis bisects BC.
Again the rad. axis cuts AC at the mid-point of YY_1, and therefore at a distance from A
$= \frac{1}{2}(AY + AY_1) = \frac{1}{2}(s - a + s) = \frac{1}{2}(b + c)$.
Similarly for AB.

388. Let the ex-circle opposite A touch AB, AC at D, E, and that opposite B touch them at F, G.
Then X is the mid-point of DF.
$\therefore\ AX = \frac{1}{2}AD - \frac{1}{2}AF = \frac{1}{2}s - \frac{1}{2}(s - c) = \frac{1}{2}c$.
Again Y is the mid-point of GE,
$\therefore\ AY = \frac{1}{2}AG + \frac{1}{2}AE = \frac{1}{2}(s - c) + \frac{1}{2}s = \frac{1}{2}(a + b)$.

389. Let the common tangent touch the two circles at Q, R, and cut the common rad. axis at O and the third circle at P, P'.
Then O is the mid-point of QR, and $OP \cdot OP' = OQ^2$.
$\therefore\ \{QR, PP'\} = -1$.

390. $\because\ PQ \cdot PQ' = PR \cdot PR'$, $\therefore$ the tangents from P to the two circles are equal, $\therefore$ PN is the rad. axis.
[The above proof does not hold if P is inside the circles; the following proof holds in all cases.
Let A be the centre of the 'Q' circle, and B of the 'R' circle.
Then $PQ \cdot PQ' = PR \cdot PR'$, $\therefore\ AP^2 - AQ^2 = BP^2 - BR^2$,
$\therefore\ AN^2 - BN^2 = AQ^2 - BR^2$, $\therefore$ PN is the rad. axis.]

391. (i) The line through the mid-points of the tangents from the point to the circle.
(ii) The perpendicular bisector of the line joining the points.
(iii) The line.
(iv) The line.
(v) The line at infinity.
(vi) Any parallel line.
(vii) Any line through their point of intersection.

392. Draw any circle through the given point to cut the circle at R, R′; let the tangent to this circle at the given point cut RR′ at P; draw PN ⊥ to the line joining the given point to the centre of the given circle. Then PN is the rad. axis.

Yes, except when the point is at the centre of the circle.

393. Let A, B be the centres of the circles, PQ, PR tangents to them from P.

Draw PS ⊥ to AB, PM ⊥ to the rad. axis; and let the rad. axis cut AB at N.

Then $PQ^2 - PR^2 = AP^2 - AQ^2 - (PB^2 - BR^2)$

$= PS^2 + AS^2 - AQ^2 - PS^2 - BS^2 + BR^2$

$= AS^2 - BS^2 - (AQ^2 - BR^2)$

$= AS^2 - BS^2 - (AN^2 - BN^2)$

($\because$ N is on the rad. axis)

$= (AS - BS)(AS + BS) - (AN - BN)(AN + BN)$

$= AB\,(AS + BS - AN - BN)$

$= 2 \,.\, AB \,.\, NS$

$= 2 \,.\, AB \,.\, MP.$

394. Call the circles A, B, C.

Let the rad. axis of A, B cut the rad. axis of A, C at P.

Then the tangent from P to B = the tangent from P to A

= the tangent from P to C.

$\therefore$ P is on the rad. axis of B, C.

$\therefore$ the three rad. axes are concurrent.

395. The common tangents at the points of contact are the rad. axes, and therefore they are concurrent.

396. The circles on AB, AC as diameters both pass through A and the foot of the altitude from A. Therefore the altitude from A is their rad. axis. Similarly for the others.

$\therefore$ the orthocentre is the rad. centre of the three circles.

397. The point of concurrence of the perpendicular bisectors of the lines joining the points, i.e. the centre of the circle through the three points.

398. The rad. axes are all parallel to one another (∵ they are all ⊥ to the common line of centres).
∴ the rad. centre is at infinity.

399. The rad. axis of the two concentric circles is at infinity.
∴ the rad. centre is at infinity.
[*Or thus:* The rad. axes of each of the two concentric circles with the third circle are parallel, ∴ etc.]

400. The rad. axis is the common tangent at the point of contact. C may be any circle touching the common tangent at the same point.

401. The common chord is the rad. axis of A, B. C may be any circle through the points of intersection of A, B. The common chord is also the rad. axis of B, C.

402. At T′ draw a line ⊥ to PT′ to cut AN at B; then B is the centre of the circle. PN is the rad. axis.

404. When R is on the radical axis.

405. From points between L and L′.
The square on the radius = $NR^2 - NB'^2 - NL^2 - NB'^2$, which is negative.

406. The radius is of zero length, ∴ the circle is a point circle.

407. The radius of the orthogonal circle is equal to the tangent from P to any circle of the coaxal system.
∵ this circle with centre P passes through all the points of contact of the tangents from P to the coaxal system.
∴ this circle cuts all the circles of the coaxal system orthogonally.

408. The orthogonal circle must cut the point-circles of the system orthogonally; i.e. must pass through L and L′ which are fixed points.
Or, from first principles, thus:
Let A be the centre of a circle of the system, PQ the tangent to it from P, and let the rad. axis cut the line of centres at N.

Then $PQ^2 = PA^2 - AQ^2 = PN^2 + AN^2 - AQ^2$.

Now, since the system is of the non-intersecting type, $AN > AQ$, $\therefore PQ > PN$.

$\therefore$ the orthogonal circle cuts the line of centres at two points, L and L′ say.

And $PL^2 = PQ^2$. $\therefore PN^2 + NL^2 = PN^2 + AN^2 - AQ^2$.

$\therefore NL^2 = AN^2 - AQ^2$ which is independent of the position of P on the rad. axis.

$\therefore$ L and L′ are the same for all positions of P on the rad. axis.

409. See Theorem 42.

410. The limiting points and common points all coincide at the point of contact.

411. Let A be the given circle, and B, C, D, circles of the coaxal system.

Let the rad. axis of A, B cut the rad. axis of the system at P.

Then the tangent from P to A = that from P to B
= that from P to C.

$\therefore$ the rad. axis of A, C passes through P.

Similarly for the rad. axis of A, D, etc.

412. Proof of Ex. 389 applies.

413. Let XY cut the rad. axis in P.

Since L is a point-circle of the system,

$\therefore PL^2 = PX \cdot PY$.

$\therefore$ PX, PL, PY are in geometrical progression.

$\therefore$ the $\perp$ distances of X, L, Y from the rad. axis are in geometrical progression.

414. Let QR be the common tangent cutting the rad. axis at P; and let L be a limiting point.

Then $PL = PQ = PR$.

$\therefore \angle QLR$ is a rt. $\angle$.

415. Let L, L′ be the limiting points; N the foot of the rad. axis; and X, Y the points at which LL′ cuts any circle of the system.

Then N is the mid-point of LL′ and $NX \cdot NY = NL^2$.

$\therefore \{XY, LL'\} = -1$.

$\therefore$ the polar of L passes through L′.

416. Let P, Q, R, ... be circles of the family.

Then A and B are on the rad. axis of P, Q.

$\therefore$ AB is the rad. axis of P, Q.

Similarly AB is the rad. axis of any other two circles of the family.

$\therefore$ the family is coaxal.

If circles are described with A, B as centres to pass through the points of contact of the tangents from A, B; these circles belong to the orthogonal coaxal system.

Now the given family will be of the non-intersecting type if the orthogonal system is of the intersecting type.

i.e. if AB < the sum of the tangents from A and B and > their difference.

417. Let L, L′ be the limiting points, A the centre of any circle of the system, r its radius.

Let PQ be a diameter of the circle PLL′, and let AP cut the circle PLL′ at R.

$\because$ the circle PLL′ passes through the limiting points,

$\therefore$ the circle with A as centre cuts it orthogonally.

$\therefore AR \cdot AP = r^2$.

And $\angle$ PRQ is a rt. $\angle$.

$\therefore$ RQ is the polar of P with respect to the 'A' circle.

Similarly the polar of P with respect to any other circle of the system passes through Q.

418. Let the tangents from T to the 'A' circle touch at P, Q, and those to the 'B' circle touch at R, S. Let PQ, RS intersect at X.

Now TP = TQ = TR = TS, ∴ P, Q, R, S are concyclic.
∴ XP . XQ = XR . XS.
∴ X is on the rad. axis of the given circles.
Also T is on their rad. axis.
∴ TX is the rad. axis and is therefore ⊥ to AB.

419. The mid-point of each common tangent lies on the rad. axis. ∴ the mid-points of the four common tangents are collinear.

420. The common chord of any pair of the circles is their rad. axis; and the three rad. axes are concurrent. ∴ the three common chords are concurrent.

421. Draw the common tangents at A, B, C; they all pass through one point, S say.
Now SA = SB = SC.
∴ S is the circumcentre of △ABC.
Also SA ⊥ to EF, etc.
∴ S is the in-centre of △DEF.
∴ the circum-circle of ABC is the in-circle of DEF.

422. (i) When the coaxal system is of the intersecting type, all the circles pass through the same two points, so the problem reduces itself to the following:—To describe a circle to pass through two given points and touch a given circle (see Ex. 350).

(ii) The rad. axes of the given circle and the circles of the system all cut the rad. axis of the system at the same point (see Ex. 411).

Draw any circle of the system (it is convenient for it to cut the given circle), draw the rad. axis of this circle and the given circle and let it cut the rad. axis of the system at T; from T draw TP a tangent to the given circle; draw PC ⊥ to TP to cut the line of centres at C; with C as centre and CP as radius describe a circle—this is the required circle.

For it touches the given circle; and, since its centre is on the line of centres and it cuts orthogonally the circle of centre T (which is of the orthogonal system), it is a circle of the given system.

423. Let L, L′ be the given points, and N the mid-point of LL′.
Let one of the circles cut LL′ at A, B.
Then $\{LL', AB\} = -1$. $\therefore$ $NA \cdot NB = NL^2$.
$\therefore$ the circle is one of the system of coaxal circles of which L, L′ are the limiting points.
Similarly all the other Apollonius circles are members of that coaxal system.

424. Let T be the given point and PQ its polar. Draw $TN \perp$ to PQ. Bisect TN at O.
Then the centres of all the circles lie along TN.
Let any circle of the system that cuts PQ cut it at P′, Q′.
Then TP′, TQ′ are tangents to this circle.
And the rad. axis of this circle and the point-circle T is a line through O $\parallel$ to PQ.
$\therefore$ this circle is a member of the coaxal system of which T and N are limiting points.
Similarly every other circle of the given system that cuts PQ is a member of the coaxal system.
Any circle of the given system that does not cut PQ has PQ outside and therefore T inside; and N will be the pole of the parallel to PQ through T.
$\therefore$ by the above this circle will be a member of the coaxal system of which N and T are limiting points.
$\therefore$ all the circles of the given system are coaxal, and their rad. axis is midway between the given point and line.

425. Let L, L′ be the limiting points of the coaxal system determined by the two given circles.
Each of the common tangents subtends a rt. $\angle$ at L and L′ (see Ex. 414).

$\therefore$ the four circles whose diameters are the common tangents all pass through L and L′.

$\therefore$ they are coaxal, LL′ being their rad. axis.

Or thus: Each of the four circles is orthogonal to the two given circles. $\therefore$ the four circles belong to the associated coaxal family.

426. See Ex. 425. Since the common tangent is a diameter of the new circle, it is obvious that it cuts the given circles orthogonally.

427. Let A, B be the centres of the given circles, and r, R their radii.

Let P, Q be the centres of similitude, N the foot of the rad. axis.

Let $\overline{NA} = a$, $\overline{NB} = b$, $\overline{NP} = p$, $\overline{NQ} = q$.

Then $$a^2 - r^2 = b^2 - R^2 = t^2 \text{ say.} \quad \ldots\ldots\ldots\ldots\text{(i)}$$

Also $$\frac{a-q}{q-b} = \frac{p-a}{p-b} = \pm\frac{r}{R} \text{ (according to figure).}$$

$\therefore$ p, q are roots of the equation

$$\left(\frac{x-a}{x-b}\right)^2 = \frac{r^2}{R^2},$$

or $x^2(R^2 - r^2) + x(\quad) + a^2R^2 - b^2r^2 = 0,$

or $x^2(R^2 - r^2) + x(\quad) + a^2(R^2 - r^2) - r^2(b^2 - a^2) = 0,$

or $x^2(R^2 - r^2) + x(\quad) + t^2(R^2 - r^2) = 0.$ [by (i).]

$\therefore$ $pq = t^2$.

$\therefore$ circle on PQ as diameter is coaxal with the given circles.

The following proof is interesting but it is not applicable to the case in which one circle is inside the other.

Let A, B be the centres of the circles; and let QR, one of their common tangents, cut their rad. axis at P and the line of centres at O. Then O is one of their centres of similitude; let O′ be the other. Let the circle on OO′ as diameter cut OP again at S. Let N be the foot of the rad. axis.

Now $\angle OSO'$ is a rt. $\angle$. $\therefore$ AQ, O'S, BR are all parallel.
And $\{AB, OO'\} = -1$. $\therefore$ $\{QR, OS\} = -1$.
And P is the mid-point of QR. $\therefore$ $PS \cdot PO = PR^2$.
$\therefore$ P is on the rad. axis of the circle OSO' and the given circles. And the centres of the circles are collinear.
$\therefore$ the circles are coaxal.

428. (i) See Ex. 359.

(ii) Let A be the given pole and B the point of concurrence of the polars. Let C be the centre of one of the circles, P a point on it, and let CA cut the polar of A at N. Let O be the mid-point of AB.

The circle on AB as diameter passes through N; and, since $CN \cdot CA = CP^2$, this circle cuts the 'C' circle orthogonally. $\therefore$ $CO^2 = OA^2 + CP^2$.

Now the square on the tangent from O to the 'C' circle $= CO^2 - CP^2 = OA^2$ which is the same for each of the three given circles.

$\therefore$ the line through O $\perp$ to the line of centres is the rad. axis of the three circles, and the circles are coaxal.

429. Let A be the given pole and B the point of concurrence of the polars.

In Ex. 428, we proved that the circle on AB as diameter cuts each of the given circles orthogonally.

And there is only one circle which does this, unless they are coaxal.

$\therefore$ the locus of A is the common orthogonal circle.

430. (i) See Ex. 414.

(ii) Let CD be the internal common tangent to which OL is to be proved parallel. Let CD cut the rad. axis at E and AB at F.

Now $FB = FD$, $\therefore$ $AF + FB = FC + FD$, $\therefore$ $2OA = 2FE$ (O, E are mid-points of AB, CD). But $OL = OA$.
$\therefore$ $OL = FE$.

Again $EL = EC = EF - CF = OA - FA = OF$,

$\therefore$ OLEF is a parallelogram.

$\therefore$ OL is $\parallel$ to CD.

Similarly OL′ is $\parallel$ to the other internal common tangent.

431. (i) See Ex. 340.

(ii) By the first part the three circles are coaxal, their rad. axis being the line joining the two common points.

[If we can prove that the circumcircle cuts these circles orthogonally, the circumcentre must be on the rad. axis.]

Let the bisectors of $\angle A$ cut the base at X, Y, and let O be the mid-point of XY.

Then $\{BC, XY\} = -1$, $\therefore$ $OB \cdot OC = OX^2 = OA^2$, $\therefore$ the circle AXY cuts the circumcircle orthogonally.

Similarly the other two circles cut the circumcircle orthogonally.

$\therefore$ the circumcentre is on the rad. axis of these three coaxal circles.

$\therefore$ the two common points and the circumcentre are collinear.

432. Let A, B be the given points, and C the centre of the given circle.

Describe any circle through A, B to cut the given circle at E, F; let EF, AB intersect at O; draw OC to cut the given circle at G, H.

Now $OG \cdot OH = OE \cdot OF = OA \cdot OB$.

$\therefore$ A, B, G, H are concyclic and this is the required circle.

[If AB, EF are $\parallel$, then AB, GH are $\parallel$.]

433. Let A, B be the centres of the given circles, P the centre of the cutting circle; and let this circle cut the 'A' circle at X, Y, the 'B' circle at Z, W and AB at Q, R. Draw $PN \perp$ to AB.

Now A is the mid-point of XY, $\therefore$ $\angle$ PAX is a rt. $\angle$.
$\therefore$ $PA^2 - PB^2 = BZ^2 - AX^2 = \text{const.}$
$\because$ $AN^2 - BN^2 = PA^2 - PB^2 = \text{const.}$
$\therefore$ N is a fixed point.
Again $PQ^2 = PX^2 = PA^2 + AX^2$.
$\therefore$ $PN^2 + NQ^2 = PN^2 + AN^2 + AX^2$.
$\therefore$ $NQ^2 = AN^2 + AX^2 = \text{const.}$
$\therefore$ Q is a fixed point.
Similarly R is a fixed point.

434. Let the circle through C cut BC at Y, and that through B cut BC at Z; and let the circles intersect at X.
Now $BA^2 = BC \cdot BY$, and $CA^2 = CB \cdot CZ$.
$\therefore$ $BA^2 : AC^2 = BY : ZC$.
Again $DC \cdot DY = DA \cdot DX = DB \cdot DZ$.
$\therefore$ $BD : DC = DY : ZD = BD + DY : DC + ZD = BY : ZC$
$= BA^2 : AC^2$.

435. Let A be the centre of the given circle. Draw QM $\perp$ to the rad. axis.
Now $T^2 = QP^2 - QK^2 = 2 \cdot AK \cdot QM$ (Ex. 393),
and $T'^2 = QP^2 - QK'^2 = 2AK' \cdot QM$.
Again $AK \cdot AK' = AP^2$ (Ex. 415).
$\therefore$ $\triangle$s APK, AK'P are similar.
$\therefore$ $PK : PK' = AK : AP = AP : AK'$.
$\therefore$ $PK^2 : PK'^2 = AK : AK'$.
But $T^2 : T'^2 = AK : AK'$.
$\therefore$ $T : T' = PK : PK'$.

436. Let C be the centre of the given circle through A, A'; and D the centre of the circle PP'Q.
Draw PM $\perp$ to their rad. axis.
Then $PA^2 - PP^2 = 2PM \cdot CD$ (Ex. 393),
and, since O is a circle of the system,
$$PO^2 - PP^2 = 2PM \cdot OD \text{ (Ex. 393)}.$$
$$\therefore \frac{PA^2}{PO^2} = \frac{CD}{OD}.$$

Similarly $$\frac{P'A'^2}{P'O^2} = \frac{CD}{OD}.$$

$$\therefore \frac{PA}{PO} = \frac{P'A'}{P'O}.$$

Now, in $\triangle$s PAX, P′A′X, $\angle$ PAX = $\angle$ P′A′X, and $\angle$ PXA = $180° - \angle$ P′XA′,

$$\therefore \frac{PA}{PX} = \frac{P'A'}{P'X}.$$

$$\therefore \frac{PX}{P'X} = \frac{PO}{P'O}.$$

$\therefore$ OX bisects $\angle$ POP′.

CHAPTER XI.

437. The same straight line.

438. A concentric circle.

443. The point at infinity on the line.

444. The centre of inversion.

445. $2OB = OA + OC$ and $OA \cdot OA' = OB \cdot OB' = OC \cdot OC'$.

$$\therefore \frac{2}{OB'} = \frac{1}{OA'} + \frac{1}{OC'}, \quad \therefore \text{ OA′, OB′, OC′ are in H.P.}$$

$\therefore$ O, A′, B′, C′ is a harmonic range.

446. Let A, B, C, D form a harmonic range; let O be the point on the line, and A′, B′, C′, D′ the inverses of A, B, C, D with respect to O, and k the constant of inversion.

Let $OA = a$, $OB = b$, ..., $OA' = a'$, ...

$$\{AC, BD\} = -1, \quad \therefore \frac{b-a}{c-b} \Big/ \frac{d-a}{c-d} = -1.$$

Now $$\{A'C', B'D'\} = \frac{b'-a'}{c'-b'} \Big/ \frac{d'-a'}{c'-a'} = \frac{\frac{k^2}{b} - \frac{k^2}{a}}{\frac{k^2}{c} - \frac{k^2}{b}} \Big/ \frac{\frac{k^2}{d} - \frac{k^2}{a}}{\frac{k^2}{c} - \frac{k^2}{d}}$$

$$= \frac{b-a}{c-b} \Big/ \frac{d-a}{c-d} = -1.$$

447. B is the mid-point of AC.
$\therefore$ B and the point at infinity divide AC harmonically.
Now the inverse of the point at infinity is O.
$\therefore$ by Ex. 446, B′ and O divide A′C′ harmonically.

448. When the inverse is a straight line, it may be regarded as the limiting case of a circle.

450. Let OA be a diameter of $\odot$ OAP, and let A′, P′ be inverses of A, P. Then $OP \cdot OP' = OA \cdot OA'$.
$\therefore$ $\angle OA'P' = \angle OPA =$ a rt. $\angle$. $\therefore$ etc.

451. Let O be the centre of inversion, C the centre of the circle. Let OC cut the circle again at A and its inverse at A′; let C′ be the inverse of C. Then $OC \cdot OC' = OA \cdot OA'$, but $OA = 2OC$, $\therefore$ $OC' = 2OA$, and OA′ is $\perp$ to the inverse of the circle.

452. Since O is on the radical axis of α and β,
$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = OD \cdot OD',$$
$\therefore$ A′, B′, C′, D′ are the inverses of A, B, C, D with regard to O. But ABCD is a straight line.
$\therefore$ A′, B′, C′, D′ lie on a circle through O.

453. (i) O, P, P′ are each equidistant from A and B, $\therefore$ OPP′ is a straight line.

(ii) Let AB cut PP′ at N. Since PBP′A is a rhombus, $AN = NB$ and $\angle PNA$ is a rt. $\angle$.
$\therefore$ $OP \cdot OP' = ON^2 - PN^2 = OA^2 - PA^2 =$ const.

454. The locus of P is a circle through O; and, by Ex. 453, P′ is the inverse of P with regard to O. $\therefore$ the locus of P′ is a straight line.

455. Let O be the centre of inversion, OT a tangent to the circle, P any point on the circle, and let OP cut the circle again in P′. Take OT as radius of inversion.
Now $OP \cdot OP' = OT^2$, $\therefore$ P′ is the inverse of P, $\therefore$ the inverse of any point on the circle lies on the same circle.

456. (i) Yes; with a negative constant of inversion equal to the rectangle contained by the segments of any chord through the point.

(ii) No; for the circle then inverts into a straight line.

457. Invert with regard to their radical centre C, the constant of inversion being the product of the segments of any chord through C of either of the circles. Since the constant is the same whichever circle we take, by Exs. 455, 456, each circle will invert into itself.

458. Let O be the centre of inversion, AB the diameter through O of the given circle and C the centre of the given circle; let A′, B′, C′ be the inverses of A, B, C.

Let OP be a tangent to the given circle, then OP touches the inverse circle at P′.

Now $OP \cdot OP' = OC \cdot OC'$, $\therefore$ C, P, P′, C′ are concyclic, $\therefore \angle OC'P' = \angle OPC =$ a rt. $\angle$. $\therefore$ C′ is on the polar of O with respect to the inverse circle.

Otherwise thus: AB is divided harmonically by C and the line at infinity. $\therefore$ by Ex. 446, A′B′ is divided harmonically by C′ and O. $\therefore$ C′ is on the polar of O with respect to the inverse circle.

459. This follows at once by p. 69.

460. Let OP, PQ be the two given straight lines. Since one of the lines inverts into a straight line, the centre of inversion must be on one of the lines, at O say.

Let P′, Q′ be the inverses of P, Q.

Now the inverse of PQ is a circle through O.

$\therefore$ one of the angles between OP′ and the tangent to this circle at $P' = \angle OQ'P' = \angle OPQ$ for P, P′, Q, Q′ are concyclic ($OP \cdot OP' = OQ \cdot OQ'$).

461. Let PQ, PR be the two straight lines and O the centre of inversion.

The tangent at O to the inverse of PQ is $\parallel$ to PQ, and the tangent at O to the inverse of PR is $\parallel$ to PR.

$\therefore$ the angles between these tangents are equal to the angles between PQ and PR.

$\therefore$ the angles at which the circles intersect are equal to the angles at which the lines intersect.

462. Let O be the centre of inversion, OT a tangent to the given curve at T, and T′ the inverse of T.

The angle between OT and the curve is equal to the angle between their inverses, i.e. is equal to the angle between OT and the inverse curve.

But the angle between OT and the curve is zero.

$\therefore$ the angle between OT and the inverse curve is zero.

$\therefore$ OT′ touches the inverse curve at T′.

463. A′B′C′ is a straight line, O a point outside it; circles are described through O, A′, C′ and through O, B′, C′; then the angles at which OA′ cuts the circle OA′C′ are equal to the angles at which OB′ cuts the circle OB′C′.

464. Let OA be the diameter, and P any point on the circle. Let A′, P′ be the inverses of A, P. Inverting the theorem, we have A′P′ at rt. $\angle$s to OA′; a circle is described through O, A′, P′; then OP′ cuts the circle OP′A′ at the same angle that OP cuts PA, i.e. OP′ cuts the circle at rt. $\angle$s, $\therefore$ OP′ is a diameter.

In fact, we have the original theorem in a converse form.

465. Inverting the theorem with regard to O, we get the theorem "OP′, OQ′ are lines through a fixed point O inclined at a constant angle and intersecting a fixed circle through O in P′, Q′; the envelope of the line P′Q′ is another circle."

The inverse theorem is true, for $\angle$ P′OQ′ is constant, $\therefore$ the length P′Q′ is constant, $\therefore$ P′Q′ is at a fixed distance from the centre of the circle OP′Q′.

$\therefore$ the original theorem is true.

466. Invert with regard to A. Let B′, C′, D′ be the inverses of B, C, D.

The circles ABC, ABD cut orthogonally, $\therefore$ B′C′, B′D′ cut orthogonally.

Now the circles CAD, CBD invert into the line C′D′ and the circle C′B′D′; and, since ∠ C′B′D′ is a rt. ∠, ∴ C′D′ is a diameter of the circle C′B′D′, and ∴ cuts it orthogonally.
∴ the circles CAD, CBD cut orthogonally.

467. Invert with regard to O; let A′, B′, C′ be the inverses of A, B, C. Write corresponding properties in parallel columns.

O, A, B, C are concyclic,	A′, B′, C′ are collinear,
a circle is described on OA as diameter.	a st. line is drawn through A′ at rt. ∠s to OA′.
etc.	etc.
To prove that these circles intersect again in three collinear pts.	To prove that these st. lines intersect again in three pts. which are concyclic with O.

Proof of inverse theorem. The feet of the perpendiculars from O upon the three straight lines are collinear, ∴ O is on the circumcircle of the triangle formed by these three lines.
∴ the given theorem is true.

468. Inverting with regard to O, and taking B′, C′, E′, F′ for the inverses of B, C, E, F, we get the inverse theorem: if B′C′, B′E′, C′F′ are three st. lines, and OB′F′, OC′E′ are st. lines ⊥ to C′F′, B′E′, then B′E′, C′F′ intersect on the ⊥ from O to B′C′.
The inverse theorem follows at once from the fact that O is the orthocentre of the triangle whose sides are B′C′, B′E′, C′F′. ∴ the given theorem is true.

469. Let the line OPQ cut ⊙PQR at P, Q and the polar of O at S.
Inverting with regard to O, ⊙PQR inverts into a ⊙P′Q′R′, the straight line OPQ into a straight line OP′Q′, and the polar of O with respect to ⊙PQR into a circle having for diameter the line joining O to the centre, C, of ⊙P′Q′R′; this circle will cut OP′Q′ at S′ the inverse of S.

Now $\angle OS'C$ is a rt. $\angle$, $\therefore$ S is the mid-point of $P'Q'$.
$\therefore$ $\{PQ, OS\}$ is harmonic (Ex. 445).

470. Let L, L′ be the limiting points, A the centre of any circle of the system, and Q any point on that circle.
Any circle through L, L′ cuts the circles of the system orthogonally. $\therefore$ $AL \cdot AL' = AQ^2$.
$\therefore$ L, L′ are inverse points with regard to any circle of the system.

471. Let O, O′ be the common points of the system.
Invert with regard to O; since all the circles of the system pass through O, they invert into straight lines; and since all the circles pass through O′, the straight lines all pass through the inverse of O′.

472. "If a system of straight lines are concurrent, a system of circles can be described which cut all the straight lines orthogonally."
It is obvious that the circles are concentric, the centre being the point of concurrence of the lines.

473. Invert with regard to a limiting point; the system of circles through the limiting points inverts into a system of concurrent lines through the inverse of the other limiting point, and these lines must be cut orthogonally by the circles inverse to the given circles.
$\therefore$ the given circles invert into a system of concentric circles having the inverse of the other limiting point for centre.

474. Another system of intersecting circles, the common points being the inverses of the common points of the given system.

475. Consider the orthogonal system; it inverts into another system of intersecting circles (Ex. 474) which must cut orthogonally all the circles inverse to the circles of the given system.

$\therefore$ the given system inverts into another system of non-intersecting coaxal circles.

476. The circle may be regarded as the intersection of a sphere and a plane.

The sphere and the plane invert into two spheres.

$\therefore$ the circle inverts into the intersection of two spheres, i.e. into another circle.

N.B. The given circle and its inverse are not, in general, in the same or parallel planes; this can be easily seen by regarding the given circle as the intersection of a plane and a sphere through the centre of inversion.

477. [Misprint; the last sentence should read "Describe a sphere through O and the circumference of the given circle; join O to the pole of the plane of the circle with respect to this sphere; this line cuts the sphere at P."]

Let T be the pole of the plane of the circle.

Take a section through O, T, and the centre of the sphere.

Let A, B be the section of the given circle, and let AB cut the tangent at O in X.

Then the polar of T passes through X, $\therefore$ the polar of X passes through T, and it also passes through O, $\therefore$ OT is the polar of X, $\therefore$ O{AB, XP} $= -1$.

Let A′, B′, ... be the inverses of A, B, ...

Now the sphere inverts into a plane $\perp$ to the diameter through O, and the inverse circle lies in this plane.

And the inverse of OX cuts this plane at infinity. $\therefore$ A′P′ ∥ P′B′. By the symmetry of the figure A′B′ is a diameter of the inverse circle, $\therefore$ P′ is the centre of the inverse circle.

478. Since P, Q are inverse points, the polar of Q passes through P; let it cut AB at R.

Then P{AB, QR} $= -1$, and $\angle$ QPR $= 90°$.

$\therefore$ PQ bisects $\angle$ APB.

479. *The following special cases admit of easy proofs.*

(i) If the circle of inversion cuts the given circle, at X, Y, say, the inverse circle passes through X, Y, ∴ the three circles are coaxal.

(ii) If the given circle and its inverse are not one inside the other, draw QQ′ a common tangent to these two circles, and let it cut the circle of inversion at R, R′.

Now QQ′ passes through O the centre of inversion, and $OQ \cdot OQ' = OR^2$. ∴ $\{QQ', RR'\} = -1$.

∴ the tangents to the three circles from the mid-point of QQ′ are all equal.

But the centres of the circles are collinear.

∴ the three circles are coaxal.

General proof.

Let O be the centre of A, the circle of inversion; let B, C be the given circle and its inverse.

Let P be any point on B, and P′ its inverse with respect to A.

Any circle through P, P′ cuts A orthogonally; describe a circle, D, through P, P′ to cut B orthogonally.

Inverting we see that D inverts into itself, ∴ D, C cut orthogonally.

∴ A, B, C have their centres collinear, and have a common orthogonal circle. ∴ A, B, C are coaxal.

480. Three given circles have, in general, one common orthogonal circle. If the three circles are inverted with regard to any point on this circle, the inverse circles are cut orthogonally by a straight line (the inverse of the orthogonal circles). ∴ the centres of the inverse circles are collinear.

481. Let A, B be the centres of the orthogonal circles, C one of their points of intersection; draw CN ⊥ to AB.

Since ∠ ACB is a rt. ∠, ∴ $AN \cdot AB = AC^2$ and $BN \cdot BA = BC^2$.

∴ etc.

482. Let P, Q be inverse points with regard to a circle whose centre is C.

Invert with respect to a circle whose centre is O, and let P′, Q′, ... be the inverses of P, Q, ..., and let D be the centre of the inverse of the 'C' circle.

Now ⊙OPQ cuts the 'C' circle orthogonally.

∴ P′Q′ is a straight line which cuts the inverse of the 'C' circle orthogonally, and ∴ passes through its centre.

Again CPQ is a straight line, ∴ O, C′, P′, Q′ are concyclic, and C′, O are inverse points with regard to the 'D' circle (Ex. 458).

∴ DP′ . DQ′ = DC′ . DO = (radius)².

∴ P′, Q′ are inverse points with regard to the 'D' circle.

483. ∠s PBD, PCD are rt. ∠s. ∴ P, B, C, D are concyclic.

∴ ∠ PDB = ∠ PCA.

Similarly ∠ PCA = ∠ PEA.

∴ ∠ PDB = ∠ PEA. ∴ P, D, E, F are concyclic.

The new theorem is:

"If P, A′, B′, C′ are concyclic points, and A′F′E′P, B′F′D′P, C′E′D′P are circles on PA′, PB′, PC′ respectively as diameters, then D′, E′, F′ lie on a straight line."

484. The angle at which the circles intersect is unaltered by inversion, ∴ the angle between the radii to one of their common points is unaltered.

∴ the cosine of that angle is unaltered.

∴ $\frac{r^2 + r'^2 - c^2}{2rr'}$ is unaltered.

∴ $c^2 - r^2 - r'^2 : rr'$ is unaltered.

485. Inverting with regard to O, we get two straight lines intersecting at P′, and parallels to the lines through O cutting them at A′, B′. The circle OAB inverts into the straight line A′B′, and Q′, the inverse of Q, is the point of intersection of A′B′, OP′. Since OA′P′B′ is a parallelogram, OP′ = 2OQ′. ∴ OQ = 2OP.

Again let the parallel to the tangent at O to OAB cut OA, OB at C, D.

Then it is easy to show that CD is parallel to A′B′; and OP bisects A′B′. ∴ P is the mid-point of CD.

486. Invert with regard to O, and let A′, B′, ... be the inverses of A, B, ...

Then O, A′, B′, C′ are concyclic points; P′, Q′, R′ are the images of O in B′C′, C′A′, A′B′.

Now the mid-points of OP′, OQ′, OR′ determine the Simson line of O with respect to △A′B′C′. ∴ P′, Q′, R′ are collinear. ∴ O, P, Q, R are concyclic.

The circle through O, A orthogonal to circle OBC inverts into the altitude from A′ to B′C′, similarly the other two circles invert into the other two altitudes of △A′B′C′; these altitudes all pass through one point, H′ say.

∴ these three circles pass through one point.

Since OH′ is bisected by the Simson line of O (Ex. 115), ∴ H′ lies on P′Q′R′.

∴ the common point of these three circles lies on the circle OPQR.

487. S and P are inverse points with regard to ⊙ DEF.

∴ the locus of S is the inverse of the locus of P.

∴ the locus of S is another circle; except when ⊙ABC passes through the centre of ⊙ABC, in which case the locus is a straight line.

488. Let O, X be the common points of the two circles, O being on PQ. Let R be the mid-point of PQ.

Invert with regard to O, and let P′, Q′ ... be the inverses of P, Q, ...

Since R is the mid-point of PQ, {P′Q′, OR′} = − 1 (Ex. 445).

And P′, Q′ move on fixed straight lines through X′.

∴ R′ moves on a fixed straight line through X′.

∴ R lies on a circle through O, X.

Again the centres of the three circles invert into the images of O in X′P′, X′Q′, X′R′.

∴ the lines joining O to the centres of the circles with the perpendicular to OX′ through O form a harmonic pencil (for X′ {P′Q′, OR′} = − 1).

And the ⊥ to OX′ is parallel to the line of centres.

∴ the centre of the circle on which R lies is midway between the centres of the given circles.

489. If the coaxal system is of the intersecting type, invert with regard to a common point; the coaxal system gives us a system of concurrent lines, and the given circle becomes a circle (or straight line).

Now one of the concurrent lines must pass through the centre of this circle (or cut the straight line at rt. angles).

∴ one of the concurrent lines cuts the inverse of the given circle orthogonally; but, if the vertex of the pencil is the centre of the circle, all the concurrent lines cut that circle orthogonally.

∴ one circle of the coaxal system cuts the given circle orthogonally; but in the case in which the common points of the given system are inverse points with regard to the given circle, all the circles of the system cut the given circle orthogonally.

Again, if the coaxal system is of the non-intersecting type, invert with regard to a limiting point; the coaxal system gives us a system of concentric circles, and the given circle becomes a circle or straight line.

Now one of the concentric circles cuts the inverse of the given circle orthogonally, unless this inverse of the given circle encloses the centre of the concentric circles or is a straight line not passing through the centre.

∴ the given circle is cut orthogonally by one circle of the coaxal system unless the given circle encloses or passes through one and one only of the limiting points. (If the given circle passes through both limiting points it is cut orthogonally by every circle of the system.)

490. Invert with regard to any point on the circle which is to be cut orthogonally; we have to show that circles which cut a given line orthogonally (i.e. have their centres on a given line) and cut a given circle at a given angle also cut another fixed circle at the same fixed angle.

It is obvious that this is true, the second fixed circle being the image of the given circle in the given line.

$\therefore$ the original theorem is true.

491. Consider the figure consisting of A, B, C, D and two circles which satisfy the given conditions. Invert with regard to A; we have two circles cutting at the given angle, one with its centre at B′ (Ex. 458), and the other having C′, D′ as inverse points (Ex. 482).

Now we have to show that in the inverse figure it is possible in an infinite number of ways to draw two circles cutting at the given angle such that B′ is the centre of one of the circles and C′, D′ are inverse points with regard to the other.

It is clear that an infinite number of circles may be drawn having C′, D′ as inverse points. Let S be any one of these circles. With centre B′ a circle can be drawn to cut S at the given angle. $\therefore$ etc.

492. $\triangle$s OPQ, OQ′P′ are similar.

$\therefore$ $PQ : P'Q' = OP : OQ' = OP \cdot OQ : OQ \cdot OQ' = OP \cdot OQ : k^2$.

493. Let P′, Q′, ... be the inverses of P, Q, ...

By Ex. 492, $\dfrac{P'A'}{PA} = \dfrac{OP' \cdot OA'}{k^2}$, $\dfrac{A'Q'}{AQ} = \dfrac{OA' \cdot OQ'}{k^2}$, etc.

Hence the proposition becomes: If P′, Q′, R′, S′ are points on a straight line, O a point outside the line, and if circles P′A′Q′, R′A′S′ both pass through O, then

$$\frac{P'A'}{OP' \cdot OA'} \cdot \frac{A'Q'}{OA' \cdot OQ'} = \frac{R'A'}{OR' \cdot OA'} \cdot \frac{A'S'}{OA' \cdot OS'},$$

$$\text{i.e. } \frac{P'A' \cdot A'Q'}{OP' \cdot OQ'} = \frac{R'A' \cdot A'S'}{OR' \cdot OS'}.$$

494. Let BO cut AC′ in N.
By similar △s OAC′, OCA, $\angle OAC' = \angle OCA = \frac{1}{2}C$.
And $\angle AON = \frac{1}{2}A + \frac{1}{2}B$.
∴ ∠ ANO is a rt. ∠.
Similarly C′O is ⊥ to AB′.
∴ O is the orthocentre of △ AB′C′.

495. If we invert with regard to O, the two given circles become two straight lines, the third circle becomes a circle, C, touching the two straight lines, and the point inverse to O with regard to the third circle becomes the centre of C (Ex. 458).
∴ the given theorem is the inverse of the theorem : "If two intersecting straight lines are touched by a variable circle, the locus of the centre of the circle consists of two straight lines at right angles to one another."
∴ the given theorem is true.

496. Let O be the centre of similitude which is to be the centre of inversion, O′ the other centre of similitude.
Let OPP′ touch the circles at P, P′; and let OPP′ cut the rad. axis at T.
Draw O′N ⊥ to OPP′.
T is the mid-point of PP′; and $\{PP', ON\} = -1$ (O, O′ divide the line joining the centres harmonically).
∴ ON . OT = OP . OP′. (See Ex. 264.)
∴ N is the inverse of T.
∴ the rad. axis inverts into the circle through ON whose centre is on OO′; this circle obviously passes through O′.

497. (i) Let O be a centre of similitude of the two circles.
Draw any line through O to cut one of the circles in P, Q; then this line cuts the other circle at P′, the point directly corresponding to P.
∴ OP : OP′ = const., but OP . OQ = const.
∴ OP′ . OQ = const.
∴ P′ is the inverse of Q with regard to O. ∴ etc.

(ii) Take any point X on this circle of inversion, draw XY, XZ tangents to the given circles, and let the centres of the given circles be A, B.

Now invert with respect to X ; let R, r be the radii of the circles into which the given circles invert ; Y′, Z′ the inverses of Y, Z.

Since XY, XZ are tangents to these latter circles, and their centres lie on XA, XB,

$$\therefore R : AY = XY' : XY = XY \cdot XY' : XY^2.$$

Similarly $r : BZ = XZ \cdot XZ' : XZ^2$.

Now the two given circles are coaxal with the circle on which X lies. (Ex. 479.)

$\therefore XY^2 - 0 = 2OA$. (the perpendicular from X on the rad. axis). (Ex. 393.)

And $XZ^2 - 0 = 2OB$. (the perpendicular from X on the rad. axis). (Ex. 393.)

$\therefore XY^2 : XZ^2 = OA : OB = AY : BZ$.

And $XY \cdot XY' = XZ \cdot XZ'$.

$\therefore R = r$.

498. (i) The sphere will invert into a plane.

By Ex. 476 any circles on the sphere will invert into circles on the plane.

Now the meridians all pass through two fixed points, $\therefore$ the inverses of the meridians form a system of circles in the plane through two fixed points, i.e. a system of intersecting coaxal circles.

The parallels cut the meridians orthogonally,

$\therefore$ the parallels invert into a system of circles cutting the former coaxal system orthogonally, i.e. a system of non-intersecting coaxal circles.

(ii) Invert with regard to R.

The sphere inverts into a plane.

$\odot$ PQR inverts into a straight line P′Q′.

$\odot$OPQ inverts into a circle O′P′Q′ cutting P′Q′ orthogonally, i.e. a circle on P′Q′ as diameter.

$\odot$s PRO, RQO invert into straight lines P′O′, Q′O′.

Then O′P′, O′Q′ cut at rt. $\angle$s.

$\therefore$ the arcs of $\odot$s PRO, RQO cut at rt. $\angle$s at R.

499. (i) See Ex. 476.

(ii) Invert with regard to the given point.

The orthogonal circles invert into a series of straight lines which cut a circle C′ (the inverse of the given circle C) orthogonally at two points.

$\therefore$ the straight lines all pass through the centre of C′.

$\therefore$ the orthogonal circles all intersect in another common point.

(iii) Again, let P be the point that is to invert into the centre of C′ (the inverse of the given circle C).

Describe a sphere whose surface contains C and P.

Let T be the pole of the plane of C with regard to the sphere.

Let TP cut the sphere again at O.

Now any plane through TPO cuts the sphere in a circle D; let C and D intersect at Q.

Then TQ touches the sphere at Q and $\therefore$ touches $\odot$D at Q; and TQ obviously cuts $\odot$C at rt. $\angle$s.

$\therefore$ $\odot$D cuts $\odot$C orthogonally at two points and passes through a fixed point O.

Invert with regard to O; by (ii) the inverse of P will be the centre of the inverse of C.

500. A section through the axis of the anchor ring consists of two circles, images of one another in the axis.

Inverting with regard to any point on the axis of the anchor ring, we get two circles which are also images of one another in the axis.

Hence, by rotating about the axis, we see that the inverse of the anchor ring with regard to any point on its axis is another anchor ring.

Again, consider a section through the axis.

The anchor ring may be regarded as the envelope of a sphere (of varying radius) whose centre lies in the axis and which always touches the two circles of the section.

Invert with regard to one of the limiting points of the coaxal system defined by the two circles of the section.

The inverse of the anchor ring is the envelope of a sphere whose centre lies in the section and which always touches two concentric circles in the plane of the section; this envelope is another anchor ring.

Now if we rotate about the axis; the limiting point with regard to which we inverted describes a circle (whose plane is perpendicular to the axis and whose centre is on the axis).

$\therefore$ the required locus consists of a straight line and a circle.

501. Let O_1, O_2 be the centres of C_1, C_2.

Let R be a point of the given figure; R_1, R_2 the inverses of R with regard to C_1, C_2. Let O_1R_2, O_2R_1 intersect at Q.

To prove that Q is the inverse of R_1 with regard to C_2.

Find N the inverse of O_2 with regard to C_1.

Now C_1, C_2 cut orthogonally.

$\therefore$ N is the inverse of O_1 with regard to C_2.

Since $O_1R_1 . O_1R = O_1N . O_1O_2$.

$\therefore$ R_1, R, O_2, N are concyclic.

Invert this circle with regard to C_2.

$\because$ O_2 is on the circle, the inverse is a straight line.

And $\because$ $O_2N . O_2O_1 = (\text{radius of } C_2)^2 = O_2R_2 . O_2R$.

$\therefore$ the st. line O_1R_2 is the inverse of the circle R_1RO_2N.

$\therefore$ Q is the inverse of R_1 with regard to C_2.

Similarly Q is the inverse of R_2 with regard to C_1.

Similarly for all positions of R.

$\therefore$ the inverse of S_1 with regard to C_2 is the inverse of S_2 with regard to C_1.

502. See Ex. 482.

503. (i) Invert with regard to P; let A′, B′, C′, D′ be the inverses of A, B, C, D.

Then A′C′D′, B′C′D′ are equal circles; and the straight line C′D′, by symmetry, bisects the angles of intersection of these circles.

∴ ⊙ PCD bisects the angles of intersection of ⊙s ACD, BCD.

(ii) First we will prove that, if P is a point such that circles ACD, BCD, ABD invert into equal circles, then D′ is the orthocentre of △ A′B′C′.

∵ ⊙s A′B′D′, A′C′D′ are equal,

∴ ∠ A′B′D′ = ∠ A′C′D′.

Similarly ∠ B′A′D′ = ∠ B′C′D′ and ∠ C′A′D′ = ∠ C′B′D′.

But the sum of all these ∠s = 2 rt. ∠s.

∴ ∠ B′A′D′ + ∠ A′B′D′ + ∠ C′B′D′ = 1 rt. ∠.

∴ A′D′ is ⊥ to B′C′.

Similarly B′D′ is ⊥ to C′A′.

∴ A′, B′, C′, D′ form a △ and its orthocentre.

We have now to show that there are four possible positions of P.

By (i) P must lie on one of the two circles which bisect the angles of intersection of ⊙s ACD, BCD; and also on the two corresponding circles for ⊙s ABD, ACD.

Invert the figure with regard to D and let A″, B″, ... be the inverses of A, B, ... ⊙s ACD, BCD, ABD, invert into straight lines A″C″, B″C″, A″B″. P″ must lie on one of the bisectors of the angles between A″C″, B″C″, and also on one of the bisectors of the angles between A″B″, A″C″.

∴ P″ must be either the incentre or one of the excentres of △A″B″C″.

∴ P″ has four possible positions P″, Q″, R″, S″.

∴ P has four possible positions P, Q, R, S.

Again we know that P″, Q″, R″, S″ form a △ and its orthocentre.

I.e. the inverses of P, Q, R, S with regard to D form a $\triangle$ and its orthocentre.

504. If we invert with regard to the fixed point, the radius of inversion being equal to the length of the given tangent, the moving circle inverts into itself and the fixed circle inverts into another fixed circle touching the moving circle.

$\therefore$ the moving circle always touches a second fixed circle.

CHAPTER XII.

505. Lines parallel to plane of projection are unaltered. Equal lines remain equal if they are equally inclined to line of steepest slope.

506. (*a*) A trapezium with sides $\parallel$ to line of steepest slope is reduced in ratio $\cos\theta : 1$; for the parallel sides are reduced in this ratio, and the distance between them is unchanged.

(*b*) By drawing $\perp$s from the vertices of the $\triangle$ to the intersection of the planes, three trapezia are formed, say X, Y, Z; and $\triangle \equiv X + Y - Z$. Each trapezium is reduced in ratio $\cos\theta : 1$; $\therefore$ so is the triangle.

(*c*) Any rectilinear figure may be divided into triangles.

507. One arm must be a line of greatest slope.

508. Diminished if acute; increased if obtuse; for draw a line of greatest slope, and consider the tangent of the angle.

509. Increased if acute; diminished if obtuse; see Ex. 508.

510. (i) increased (unless reflex), (ii) diminished.

511. The bisector must be $\parallel$ or $\perp$ to lines of greatest slope.

512. (i) By Exx. 508, 509, one half of the rt. ∠ would be diminished, and one increased.

(ii) The square would project into a parallelogram, whose diagonal does not generally bisect the angle.

513. (i) Relation unchanged only in case of rt. angled △ with sides ∥ and ⊥ to lines of greatest slope.

(ii) As (i).

(iii) Relation always unchanged by § 9.

(iv) Relation unchanged only if base is ∥ or ⊥ to lines of greatest slope.

(v) As (i).

(vi) Always unchanged.

(vii) Unchanged if sides ∥ and ⊥ to lines of greatest slope.

(viii) Unchanged if a diagonal line of greatest slope.

(ix) Always unchanged.

(x) Always changed.

(xi) Always unchanged.

(xii) Always changed.

514. A parallelogram pattern.

515. See Ex. 513 (iii).

516. (5) Diameters of the ⊙ at rt. ∠s project into conjugate diameters of the ellipse. Lines joining ends of diameter of circle to point on ⊙ are ∥ to diameters at rt. ∠s.

(7) Let a pair of diameters of a ⊙ at rt. ∠s cut the tangent at p in t, t'; and let cd be a radius ∥ to tangent at p. Then it is easily proved that $pt.pt' = cd^2$. And

$$pt : \mathrm{PT} = pt' : \mathrm{PT}' = cd : \mathrm{CD}, \quad \therefore \mathrm{PT}.\,\mathrm{PT}' = \mathrm{CD}^2.$$

(10) Using the same letters for the circle,

$$\mathrm{O}p\,.\,\mathrm{O}p' = \mathrm{O}q\,.\,\mathrm{O}q' \text{ and } \mathrm{CP}^2 = \mathrm{CQ}^2.$$

$$\therefore \mathrm{O}p\,.\,\mathrm{O}p' : \mathrm{CP}^2 = \mathrm{O}q\,.\,\mathrm{O}q' : \mathrm{CQ}^2.$$

Now projection alters the lines $\mathrm{O}p$, $\mathrm{O}p'$, CP in the same ratio, $\therefore \mathrm{O}p\,.\,\mathrm{O}p' : \mathrm{CP}^2$ is unaltered. So also is $\mathrm{O}q\,.\,\mathrm{O}q' : \mathrm{CQ}^2$, $\therefore$ etc.

(11) As (10).

(12) Particular case of (10).

(13) Area of circle $= \pi a^2$, $\therefore$ area of ellipse $= \pi a^2 \cos\theta$; but $a \cos\theta = b$, $\therefore$ etc.

(14) Area of square circumscribing $\odot = 4a^2$; this projects into parallelogram formed by tangents at extremities of conjugate diameters of ellipse; whose area is therefore

$$4a^2 \times \frac{b}{a} = 4ab.$$

(15) Let cp, cd be conjugate semidiameters of the $\odot$; pm, dn ordinates. Then $pm = cn$, $dn = cm$. On projecting,

$$\mathsf{PM} = pm \cos\theta = cn \cos\theta, \quad \mathsf{DN} = cm \cos\theta;$$

$$\begin{aligned} \therefore\ \mathsf{CP}^2 + \mathsf{CD}^2 &= \mathsf{CM}^2 + \mathsf{PM}^2 + \mathsf{CN}^2 + \mathsf{DN}^2 \\ &= cm^2 + cn^2 \cos^2\theta + cn^2 + cm^2 \cos^2\theta \\ &= (cm^2 + cn^2)(1 + \cos^2\theta) \\ &= a^2(1 + \cos^2\theta) = a^2 + b^2. \end{aligned}$$

517. If the word "ellipse" is substituted for "circle" in Theorems 31, 32, 33, the resulting theorems are true.

518. Projecting ellipse into $\odot$, we should make $\triangle$s cpn, caq congruent, and $\angle cqa = \angle cnp =$ rt. $\angle$; $\therefore aq$ is $\parallel$ to tangent at p in circle; $\therefore$ etc.

CHAPTER XIII.

519. $\{\mathsf{AB}, \mathsf{CD}\} = -1$; $\{\mathsf{CD}, \mathsf{AB}\} = -1$; $\{\mathsf{AC}, \mathsf{BD}\} = 2$, etc. All the different ratios reduce to three, -1, 2, $\frac{1}{2}$.

520. Let $\mathsf{OA} = a$, $\mathsf{OB} = b$, etc. Then $\mathsf{OA}' = \dfrac{k^2}{a}$, etc.

$$\{\mathsf{AB}, \mathsf{CD}\} = \frac{\mathsf{AC}}{\mathsf{CB}} : \frac{\mathsf{AD}}{\mathsf{DB}} = \frac{(c-a)(b-d)}{(b-c)(d-a)},$$

$$\{A'B', C'D'\} = \frac{\left(\frac{k^2}{c} - \frac{k^2}{a}\right)\left(\frac{k^2}{b} - \frac{k^2}{d}\right)}{\left(\frac{k^2}{b} - \frac{k^2}{c}\right)\left(\frac{k^2}{d} - \frac{k^2}{a}\right)}$$

$$= \frac{(a-c)(d-b)}{(c-b)(a-d)} = \{AB, CD\}.$$

If A coincides with O, the harmonic range $\{OB, CD\}$ would invert into equicross range $\{\infty B', C'D'\}$;

$\therefore \frac{\infty C'}{C'B'} : \frac{\infty D'}{D'B'} = -1$. But $\frac{\infty C'}{\infty D'} = 1$; $\therefore C'B' = -D'B'$;

$\therefore$ B′ is mid-point of C′D′.

521. $\frac{AC}{CB} = \frac{A'C'}{C'B'}$, etc.

522. $\frac{AC}{CB} : \frac{AD}{DB} = \frac{AC}{CB} : \frac{AE}{EB}$; $\therefore \frac{AD}{DB} = \frac{AE}{EB}$; $\therefore$ D, E coincide.

523. A′B′C′D′ is divided in the same ratios as ABCD.

524. If $\{AB, CD\} = 1$, $\frac{AC}{CB} = \frac{AD}{DB}$; $\therefore \frac{AB - CB}{CB} = \frac{AB - DB}{DB}$;

$\therefore \frac{AB}{CB} = \frac{AB}{DB}$; $\therefore$ either $AB = 0$ and A, B coincide; or $CB = DB$ and C, D coincide.

If $\{AB, CD\} = 0$, either $\frac{AC}{CB} = 0$, or $\frac{AD}{DB} = \infty$.

In former case C coincides with A; in latter, D with B.

If $\{AB, CD\} = \infty$, either $\frac{AC}{CB} = \infty$; or $\frac{AD}{DB} = 0$.

Therefore either C coincides with B, or A with D.

525. $\frac{AC}{CB} \cdot \frac{DB}{AD} = \frac{AD}{DB} \cdot \frac{CB}{AC}$; $\therefore \frac{AC}{CB} \cdot \frac{DB}{AD} = \pm 1$;

$\therefore$ $\{AB, CD\}$ is harmonic, if all the points are distinct.

526. $\{AB, CC'\} = -1$; $\therefore \frac{AC}{CB} = -\frac{AC'}{C'B}$.

Also $\frac{AD}{DB} = -\frac{AD'}{D'B}$.

$\therefore \frac{AC}{CB} \cdot \frac{DB}{AD} = \frac{AC'}{C'B} \cdot \frac{D'B}{AD'}$.

529. If ABCD, A′B′C′D′ are $\parallel$, then AB : BC = A′B′ : B′C′, etc.

If ABCD, A′B′C′D′ are not $\parallel$, through B′ draw XB′Y $\parallel$ to ABC, meeting OA, OC in X, Y. Then B′X : B′Y = BA : BC = B′A′ : B′C′; and $\triangle$s XB′A′, YB′C′ are similar.

Hence AA′, CC′ are $\parallel$. Similarly all the four lines AA′, BB′, CC′, DD′ are $\parallel$.

530. O $\{AC, BD\} = \{PR, Q\infty\}$

$$= \frac{PQ}{QR} : \frac{P\infty}{\infty R} = -\frac{PQ}{QR} = \frac{PQ}{RQ}.$$

Or prove as in Th. 27.

532. (i) Two pencils of $\parallel$ rays subtended by the same range are equicross.

(ii) If XYZW is the line at ∞, we have two pencils at P, Q, with corresponding rays parallel.

533. (i) If Q is the image of P, clearly the two pencils are equiangular, (ii) if P, Q are on opposite sides of the line, let Q′ be the image of Q. Then P, Q′ are on the same side; and if P $\{XYZW\}$ = Q $\{XYZW\}$, also P $\{XYZW\}$ = Q′ $\{XYZW\}$. We may, therefore, without loss of generality, suppose that P, Q are on same side.

If the two pencils are equiangular and P, Q are on the same side of line, then PQXYZW are concyclic. We should thus have a line cutting a $\odot$ in four points, which is not possible unless the $\odot$ consists of the line PQ together with a line at ∞, XYZW.

534. (i) {ABFW}, (ii) {EXFZ}.

535. If AA′, BB′, CC′ are concurrent, either they are ∥, in which case DD′ is ∥ to them; or ABCD, A′B′C′D′ are ∥.

536. Any position except the two described in Ex. 535.

539. X′{PXYZ} = X{PX′Y′Z′}; and the two pencils have a ray XX′ common; ∴ the intersections of other pairs are collinear, namely, P, the intersection of X′Y, XY′, and that of X′Z, XZ′; ∴ etc.

540. Let QB, PC meet at X; QC, PB at X′; QB, PD at Y; QD, PB at Y′. Let PQ meet DCB in O. Then

$$\{PY'X'B\} = Q\{PY'X'B\} = \{ODCB\}$$

and $$\{QYXB\} = P\{QYXB\} = \{ODCB\};$$

∴ {PY′X′B} and {QYXB} are equicross and have B in common; ∴ XX′, YY′, PQ are concurrent; ∴ etc.

541. Let P, Q, R be the points; O the required point on the line AB. Let a parallel to AB cut OP, OQ, OR in X, Y, Z; XY : YZ is to be a given ratio. Let PR cut AB in V. Take U on PR so that {PR, UV} = {XZ, Y∞} = XY : ZY.

Since PV : VR is known, this gives PU : UR in magnitude and sense; this gives U uniquely. QU cuts AB in the required point.

542. The angles or their supplements are given, ∴ the sines of the angles are given.

543. Let O be the centre of the ⊙; let the tangents at fixed points P, Q, R, S cut the tangent at movable point X in A, B, C, D; ∠AOB = ½∠POQ, etc.; ∴ O{ABCD} is a pencil of fixed angles.

544. Let the points A, B, C, D lie in a line whose pole is P. Then the polars of A, B, C, D pass through P. Also the polar of A is ⊥ to OA, etc., ∴ the pencil of polars is equiangular to the pencil O{ABCD}.

545. Let $A_1B_1Q_1$, $A_2B_2Q_2$, $A_3B_3Q_3$ be three positions of ABQ. Then $O\{XA_1A_2A_3\} = P\{XA_1A_2A_3\} = P\{XB_1B_2B_3\} = O'\{XB_1B_2B_3\}$. And ray XOO′ is common to pencils $O\{XA_1A_2A_3\}$, $O'\{XB_1B_2B_3\}$. ∴ $Q_1Q_2Q_3$ are collinear (Th. 56); ∴ etc.

546. Let POO′ cut XA, XB in U, V.

$$O\{UA_1A_2A_3\} = P\{UA_1A_2A_3\} = P\{VB_1B_2B_3\} = O'\{VB_1B_2B_3\}.$$

And the ray POO′ is common to the two pencils, ∴ $Q_1Q_2Q_3$ are collinear. If PAB takes the position PX, then Q coincides with X, ∴ locus of Q is a str. line through X.

547. Let U_1O_1 cut U_2U_3 in P; O_1A_1 cut O_2O_3 in Q.

$$\begin{aligned} &U_1\{A_3O_1A_2A_1\} = \{U_2PU_3A_1\} \\ &= O_1\{U_2PU_3A_1\} = \{O_3U_1O_2Q\} \\ &= A_1\{O_3U_1O_2Q\} = A_1\{A_2U_1A_3O_1\} \\ &= A_1\{A_3O_1A_2U_1\}. \end{aligned}$$

And A_1U_1 is common to the two pencils

$$U_1\{A_3O_1A_2A_1\},\quad A_1\{A_3O_1A_2U_1\}.$$

∴ $A_3O_1A_2$ are collinear.

548. $F\{HK_1K_2K_3\} = F\{BL_1L_2L_3\} = G\{BL_1L_2L_3\}$
$= G\{CM_1M_2M_3\} = H\{CM_1M_2M_3\} = H\{FK_1K_2K_3\}$.

∴ K_1, K_2, K_3 are collinear. And if LGM coincides with GA, K coincides with A, ∴ etc.

Or thus: Let AC, LF intersect at X, and AK, BC at T.

Then $L\{BFGC\} = L\{AXMC\} = K\{TFHC\}$;

∴ T is fixed; ∴ K moves on a fixed str. line TA.

549. Let A, B move on fixed lines OU, OV, and let PQR meet these lines in U, V respectively.

$$Q\{UA_1A_2A_3\} = R\{UA_1A_2A_3\} = R\{VB_1B_2B_3\} = P\{VB_1B_2B_3\}.$$

And the equicross pencils $Q\{UA_1A_2A_3\}$, $P\{VB_1B_2B_3\}$ have

ray PQR in common, $\therefore C_1C_2C_3$ are collinear. And if ARB passes through O, C coincides with O; $\therefore$ etc.

Or thus: Let A, B move on lines Oa, Ob which cut PQR at a, b, and let OC cut PQR at c, and let OA cut BC at X.

$$\begin{aligned}\text{Then } O\{abcP\} &= O\{XBCP\}\\ &= A\{XBCP\}\\ &= A\{aRQP\};\end{aligned}$$

$\therefore c$ is fixed; $\therefore$ Oc is fixed; $\therefore$ etc.

550. Let the two fixed lines meet at O.

Then $Q\{OL_1'L_2'L_3'\} = Q\{OM_1M_2M_3\} = P\{OM_1M_2M_3\}$
$= P\{OL_1L_2L_3\} = Q\{OL_1L_2L_3\} = Q\{OM_1'M_2'M_3'\}$;

$\therefore$ $L_1'M_1'$, $L_2'M_2'$, $L_3'M_3'$ are concurrent; $\therefore$ etc.

551. Let I_1D meet I_2I_3 in P, I_1A meet BC in L. Then

$$\{I_2I_3,\ PA\} = \{CB,\ DL\},$$

$$\text{or } \frac{I_2P}{PI_3}\cdot\frac{AI_3}{I_2A} = \frac{CD}{DB}\cdot\frac{LB}{CL}.$$

$$\text{Call } \frac{I_2P\,.\,I_3Q\,.\,I_1R}{PI_3\,.\,QI_1\,.\,RI_2},\quad \Pi\frac{I_2P}{PI_3};\ \text{etc.}$$

$$\text{Then } \Pi\frac{I_2P}{PI_3}\times\Pi\frac{AI_3}{I_2A} = \Pi\frac{CD}{DB}\times\Pi\frac{LB}{CL}.$$

Now I_1A, I_2B, I_3C pass through I, $\therefore$ $\Pi\dfrac{AI_3}{I_2A} = +1$. (Ceva.)

And AD, BE, CF pass through H, $\therefore$ $\Pi\dfrac{CD}{DB} = +1$.

And AL, BM, CN pass through I, $\therefore$ $\Pi\dfrac{LB}{CL} = +1$.

$$\therefore\ \Pi\frac{I_2P}{PI_3} = -1,$$

$\therefore$ I_1D, I_2E, I_3F are concurrent.

552. Let I_1A meet CB in L, I_1a meet I_2I_3 in P.

$$\text{Then } \frac{I_2P}{PI_3}\cdot\frac{AI_3}{I_2A} = \frac{Ca}{aB}\cdot\frac{LB}{CL} = +\frac{LB}{CL}.$$

Now, as in Ex. 551, $\Pi \frac{AI_3}{I_2A} = +1$ and $\Pi \frac{LB}{CL} = +1$.

$\therefore \Pi \frac{I_2P}{PI_3} = 1$, and $I_1\alpha$, $I_2\beta$, $I_3\gamma$ are concurrent.

553. $\Pi \frac{BK'}{CK'} = \Pi \frac{C'K}{B'K} = 1$ ($\because$ KLM meets sides of $\triangle A'B'C'$ in K, L, M),

$\therefore$ K', L', M' are collinear.

554. From the given relation we get A', B', C' collinear. Let AX, BY, CZ meet BC, CA, AB in P, Q, R respectively.

Then $\frac{BP}{PC} \cdot \frac{A'C}{BA'} = \frac{C'X}{XB'} \cdot \frac{A'B'}{C'A'} = \frac{A'B'}{C'A'}$;

$$\therefore \Pi \frac{BP}{PC} \times \Pi \frac{A'C}{BA'} = 1;$$

$\therefore \Pi \frac{BP}{PC} = 1$ and AP, BQ, CR are concurrent.

555. Let the mid-point of XY be O; and let the distances of P, Q, R, P', Q', R' from O, reckoned algebraically, be p, q, r, p', q', r'.

Then $pp' = qq' = rr' = OX^2 = k^2$, say,

$$\{PP'QR\} = \frac{PQ}{QP'} \cdot \frac{RP'}{PR} = \frac{(q-p)(p'-r)}{(p'-q)(r-p)}$$

$$= \frac{\left(\frac{k^2}{q'} - \frac{k^2}{p'}\right)\left(\frac{k^2}{p} - \frac{k^2}{r'}\right)}{\left(\frac{k^2}{p} - \frac{k^2}{q'}\right)\left(\frac{k^2}{r'} - \frac{k^2}{p'}\right)}$$

$$= \frac{(q'-p')(p-r')}{(p-q')(r'-p')} = \{P'PQ'R'\}.$$

CHAPTER XIV.

556. $\{TU, XY\} = S\{TU, XY\} = \{PR, ZY\}$
$= T\{PR, ZY\} = \{SQ, ZX\}$, $\therefore$ etc.

557. Consider the quadrilateral PBQA. The third diagonal, DC, cuts AB at ∞; $\therefore$ PQ bisects AB.

559. Take RS $\parallel$ to AC; then PQ will bisect AC.

561. For the two internal diagonals, the harmonic property simply gives the bisection property. For the third diagonal, the line at infinity, the property may be stated thus: If parallels be drawn to the sides through the centre of a parallelogram, they form a harmonic pencil with the diagonals.

562. Since the polar of A passes through Q, $\therefore$ the polar of Q passes through A; similarly the polar of Q passes through C, $\therefore$ the polar of Q is AC, $\therefore$ etc.

563. $\because$ the polar of A passes through B, $\therefore$ the polar of B passes through A. Again, the polars of A and B both pass through C, $\therefore$ the polar of C is AB.

564. The line joining a pole to the centre is $\perp$ to its polar.

566. In a real circle, the centre cannot lie between pole and polar, $\therefore$ the orthocentre cannot lie between a vertex and the opposite side.

567. A point circle, at the right angle.

568. HBC is equilateral.

$$HB = BC = 2a, \quad HE = a, \quad \therefore r^2 = 2a^2, \quad \therefore r = a\sqrt{2}.$$

Again, since AP is the polar of B, BP is tangent at P, and $\perp$ to HP; and since PA bisects HB at rt. $\angle$s, $\angle HBP = 45^\circ$. But $\angle HBA = 30^\circ$, $\therefore \angle ABP = 15^\circ$.

569. If the polar circle cuts AC in P and Q, we have the side CA cut by the circle in P, Q, and by the polar of C in A, $\therefore$ etc.

570. Let O be the point-circle. The polar of any point A will be a line OB, $\perp$ to AO. If B be any point on OB, the polar of B is AO. Hence the rt. angled self-polar $\triangle$ AOB.

571. The polar of the centre is the line at ∞. The $\triangle$ will therefore consist of two lines through the centre at rt. $\angle$ s, together with the line at ∞.

572. Each triangle is formed by a pair of parallels equidistant from the line on opposite sides, together with any third line.

573. Since CAPQ is harmonic (Ex. 569), $\therefore$ EA . EC = $EP^2 = EQ^2$. Again, $\triangle$s EHA, ECB are similar,

$$\therefore \frac{EA}{EB} = \frac{EH}{EC}, \therefore EA \cdot EC = EH \cdot EB; \therefore EP^2 = EH \cdot EB;$$

and P (and Q) are on $\odot$ with HB as diameter. So also are F, D.

574. The $\odot$ on diameter BC passes through F, and HF . HC is the square of the tangent to it from H, the centre of the polar $\odot$. But, AFCD being concyclic,

$$HF \cdot HC = HA \cdot HD = (\text{rad. of polar } \odot)^2.$$

575. To prove that UZ is the polar of T, let UZ cut QR, PS in K, L. Then since U {TZ, QR} is harmonic, and is cut by QR in harmonic range {TK, QR}, $\therefore$ polar of T passes through K. Similarly polar of T passes through L, $\therefore$ polar of T is KL, or UZ.

576. To prove that X is the pole of YZ. {XZ, QS} is harmonic, $\therefore$ P {XZ, QS} is harmonic, $\therefore$ PX passes through the pole of PZ. Similarly RX passes through the pole of RZ, $\therefore$ X is the pole of YZ.

577. Let CA, C′A′ cut OBB′ in T, T′. Then

$$\{CAQT\} = \{C'A'QT'\}, \therefore B\{CAQB'\} = B'\{C'A'QB\}.$$

And these equicross pencils have BB′ in common, ∴ P, R, Q are collinear.

578. Let us add a fourth point D to the three A, B, C, and a fourth D′ to the three A′, B′, C′; so that ODD′ is a straight line. And let us ascertain whether ABCD, A′B′C′D′ necessarily have an axis of perspective.

If we begin by fixing the line ODD′, then D, D′ are fixed when BD, B′D′ are fixed. But BD, B′D′ may be drawn in any directions from B, B′; and if these meet at U, this point may be anywhere. It is not, then, of necessity, on the line PQR.

579. BC, B′C′ lie in the same plane OBC, and therefore intersect, say at P. So also CA, C′A′ intersect, say at Q; and AB, A′B′ intersect, say at R. Now P, Q, R are in the plane ABC, and also in the plane A′B′C′. Hence they are on the line of intersection of these planes.

As OAA′ rotates, the collinearity of P, Q, R is not disturbed.

581. Since P, Q, R are collinear,

$$A\{PQRS\} = A'\{PQRS'\}, \therefore \{PCSB\} = \{PC'S'B'\}.$$

And as these ranges have a common point P, ∴ BB′, AA′, CC′ are concurrent.

582. The centre of perspective is the centre of similitude. The axis of perspective is the line at infinity.

583. Yes; for the polygons have a centre of similitude.

584. Both centre and axis of perspective are at infinity.

585. By Menelaus, $\frac{BP}{CP} \cdot \frac{CQ}{AQ} \cdot \frac{AR}{BR} = 1.$

By Ceva, $\frac{BP'}{CP'} \cdot \frac{CQ}{AQ} \cdot \frac{AR}{BR} = -1.$

$$\therefore \frac{BP}{CP} = -\frac{BP'}{CP'}, \quad \therefore \{BC, PP'\} \text{ is harmonic.}$$

Let AD be the median, AL be ∥ to BC.

Then A{BC, DL} is harmonic.

Also X{CB, PP′} is harmonic.

And XC, XB, XP′ are ⊥ to AB, AC, AL respectively, ∴ XP is ⊥ to AD.

586. Consider the quadrilateral BDRC. The diagonal DC is divided harmonically by the other two diagonals, in A and P. But A, D, C are fixed, ∴ P is fixed.

587. Let ABCD be the quadrilateral and let O be the intersection of AC, BD, the points P, P′, Q, Q′ being on BC, AD, AB, CD. Let AD, BC meet in E; AD, CQ in U. {PP′, OQ} = (projecting from C on to AD) {EP′, AU} = (projecting from Q on to BC){EP, BC} = (projecting from D on to PP′){P′P, OQ′}.

$$\therefore \frac{PO}{OP'} \cdot \frac{QP'}{PQ} = \frac{P'O}{OP} \cdot \frac{Q'P}{P'Q'}, \quad \therefore QP' \cdot P'Q' = Q'P \cdot PQ.$$

$$\therefore (QP + PP')\,P'Q' = (Q'P' + P'P)\,PQ, \quad \therefore PQ = Q'P'.$$

588. BP, CQ meet at D, the extremity of the diameter through A. If AD meets BC in O, then O is the pole of PQ. Since the pole of PQ lies on BC, ∴ R, the pole of BC, lies on PQ.

589. The quadrilateral can be circumscribed round a ⊙, and the △ formed by the diagonals is self-polar. Its orthocentre is therefore the in-centre of the quadrilateral. (Ex. 564.)

590. Let the fixed tangents be OAB, OCD; the other tangents QAD, QBC; let the diagonals AC, BD meet in P. Then △ formed by OQ, PA, PD is self-polar. ∴ P lies on the polar of O, a fixed line.

591. (i) △GEF is self-polar, and G is the pole of EF. ∴ OG is ⊥ to EF.

(ii) Let EF cut OG, BD at N, R.
Then {BD, GR} = −1. (Theorem 57.)
And ∠GNR is a rt. ∠.
∴ NG bisects ∠DNB.
Similarly NG bisects ∠ANC. ∴ etc.

592. Let PQR be one of the △s formed by 3 sides of the quadrilateral, and let PX be one of the diagonals, X being a corner of the quadrilateral lying on QR.

The theorem may be stated as follows: If X is a point on the side QR of △ PQR, then the ⊙ on PX as diameter is orthogonal to the polar ⊙ of △PQR. Let O be the centre, r the radius of the polar ⊙. As QR is the polar of P, OP cuts QR at rt. ∠s; at N say. ∴ ⊙ on XP passes through N; and (tangent from O to this ⊙)^2 = ON . OP = r^2. ∴ the two ⊙s are orthogonal.

593. Let the area of a △ be considered + or − according as the perimeter is traversed in a clockwise or counter-clockwise direction; i.e. △ABC = − △ACB; with this convention, it is easy to prove the following lemma:

If Z is the mid-point of XY and P, Q any two points; then △PQX + △PQY = 2△PQZ.

In the case of the quadrilateral, 4△PEQ = 2△PEB + 2△PED = △AEB + △CED = ADCB. ∴ △PQE = $\frac{1}{4}$ABCD; similarly △PQF = $-\frac{1}{4}$ABCD. ∴ △PQE + △PQF = 0. ∴ PQ produced bisects EF.

594. As △s ABC, A′B′C′ are in perspective, the line joining the meet of AB, A′B′ to the meet of AC, A′C′ passes through the meet of BC, B′C′; i.e. is ∥ to BC; similarly, etc.

595. *Lemma.* If BC be divided harmonically by D, X, and P be the mid-point of DX, then PB : PC = BD^2 : DC^2. Let BD = x, DC = y, DP = a. Then $(a+x)(a-y) = a^2$.

∴ $a = xy/(x-y)$. Now PB/PC = $(a+x)/(a-y) = x^2/y^2$.

In the present case, FE is the polar of A and BC of D. ∴ X is the pole of AD, ∴ BDCX is harmonic (Th. 33).

∴ PB : PC = BD^2 : DC^2 = $(s-b)^2 : (s-c)^2$.

∴ by Menelaus, P, Q, R are collinear.

596. Using fig. 85, let PQ, RS meet TZ in A, A′; QR, SP meet JZ in B, B′; PR, QS meet TU in C, C′. Consider △TZU

and note that UA, TB, ZC′ are concurrent. As (TU, C′C) is harmonic, ∴ A, B, C are collinear (Ex. 191).

So for three other trios; each trio consisting of an A, a B, and a C.

MISCELLANEOUS EXERCISES.

597. △s DBF, DEC are similar. ∴ △DBF : △DEC = $BF^2 : EC^2$. But △DBF = △DEC, ∴ BF = EC. Hence △s BCE, CBF can be proved congruent, ∴ ∠B = ∠C, ∴ △ABC is isosceles.

598. Let P, Q be the given points through which AB, AC pass.

[∠B is given, ∴ the length AC can be determined; through Q draw AC of the right length.]

Take any points X, Y, Z on the circle such that

∠XYZ = given ∠ABC.

Draw a concentric circle to touch XZ; from Q draw a tangent to this circle to cut the given circle at A, C; join AP and let it cut the given circle at B. ABC is the required triangle.

For AC = XZ, and ∠ABC = ∠XYZ = given ∠.

[In general there are four solutions, for two tangents can be drawn from Q to the inner circle and either end of each tangent may be taken as A.]

599. ∠AEB = ⌣ between ABC and the tangent at A to ⊙ABE.

∠AFC = ∠ between ABC and the tangent at A to ⊙ACF.

∴ the angle at which the circles intersect = ∠AEB ~ ∠AFC = ∠ between BE and CF = const.

600. Using notation of fig. 17,

$CY_2 = s - a = BZ_3$, $AZ_3 = s - b = CX_1$, $BX_1 = s - c = AY_2$,

∴ $BX_1^2 + CY_2^2 + AZ_3^2 = CX_1^2 + AY_2^2 + BZ_3^2$. ∴ the perpendiculars to the sides at X_1, Y_2, Z_3 are concurrent by the converse proved in Ex. 129.

601. Consider two ⊙s intersecting at O, O′. POQ is drawn to cut the ⊙s at P, Q, and the tangents at P, Q meet at C.

Then $\angle CPQ = \angle PO'O$, $\angle CQP = \angle QO'O$.

$\therefore \angle PO'Q = \angle CPQ + \angle CQP = 180^\circ - \angle PCQ$.

$\therefore$ P, C, Q, O′ are concyclic.

Taking three ⊙s, let the tangents at P, Q, R form $\triangle ABC$.

$\because$ P, C, Q, O′ are concyclic, $\therefore \angle PO'C = \angle PQC$.

And $\because$ P, B, R, O′ are concyclic, $\therefore \angle PO'B = \angle PRB$.

$\therefore \angle BO'C = \angle PRB - \angle PQC = \angle BAC$.

$\therefore$ A, B, C, O′ are concyclic.

602. (i) Draw the circumcircle of ABC; produce AB to D and make BD = AB; through D draw a straight line ∥ to BC cutting the circle in Q, R; join AQ cutting BC at P.

$AP = PQ$ and $BP \cdot CP = AP \cdot PQ = AP^2$.

Another position of P is the point of intersection of AR, BC; so that, in general, there are two solutions.

(ii) Draw AX to touch ⊙ABC at A and to cut BC in P; then $AP^2 = BP \cdot CP$.

603. Let α, β, γ be the mid-points of BC, CA, AB; let Pα, Qβ intersect at X, and Qβ, Rγ at Y. Let $AP : PD = n : 1$.

$\because AP : PD = BQ : QD$, $\therefore$ PQ is ∥ to AB, and

$\therefore PQ : AB = 1 : n + 1$.

Now $\alpha\beta$ is ∥ to AB and $= \frac{1}{2}$AB. $\therefore PQ : \alpha\beta = \frac{1}{2} : n + 1$.

And $\alpha\beta$ is ∥ to PQ. $\therefore PX : X\alpha = QX : X\beta = PQ : \alpha\beta = \frac{1}{2} : n + 1$.

Similarly $\beta\gamma$ is ∥ to QR and $QY : Y\beta = RY : Y\gamma = \frac{1}{2} : n + 1$.

$\therefore QX : X\beta = QY : Y\beta$. $\therefore$ X and Y coincide.

$\therefore$ Pα, Qβ, Rγ are concurrent.

Otherwise thus: $\triangle PQR$ is similar and situated similarly to $\triangle ABC$, and therefore to $\triangle\alpha\beta\gamma$, formed by joining the mid-points of BC, CA, AB. $\therefore$ corresponding sides of $\triangle PQR$ and $\triangle\alpha\beta\gamma$ are parallel, and the two $\triangle$s have the line at infinity as an axis of perspective.

$\therefore \alpha$P, βQ, γR are concurrent.

604. The mid-points of the sides of each triangle lie on its nine-points circle.

The nine-points centre of each triangle is the mid-point of the line joining their circumcentre and orthocentre, and is therefore a fixed point. Also the radius of the nine-points circle is half the radius of the circumcircle, and is therefore constant.

$\therefore$ the nine-points circles of all the triangles coincide.

$\therefore$ the locus of the mid-points of the sides of all the triangles is a circle.

605. Q is the orthocentre of $\triangle$ PBC, $\therefore \angle$ BMC is a rt. $\angle$,

$\therefore$ the locus of M is the circle on BC as diameter.

Again ND is a diameter of the 9-points circle of $\triangle$ PBC, and M is on that circle, $\therefore \angle$ NMD is a rt. $\angle$, $\therefore$ NM touches the locus of M.

606. Let XY cut BC at Z.

Now AC is $\parallel$ to DF, $\therefore$ ZC : ZD = ZA : ZY,

and AB is $\parallel$ to DE, $\therefore$ ZB : ZD = ZA : ZX,

$\therefore$ ZC : ZB = ZX : ZY,

$\therefore$ BY is $\parallel$ to CX.

[If XY is $\parallel$ to BC, BD = AX and DC = AY, $\therefore$ XY = BC and XY is $\parallel$ to BC, $\therefore$ BY is $\parallel$ to CX.]

607. $\dfrac{BD_1}{CD_1} \cdot \dfrac{CE_2}{AE_2} \cdot \dfrac{AF_3}{BF_3} = -\dfrac{s-c}{s-b} \cdot \dfrac{s-a}{s-c} \cdot \dfrac{s-b}{s-a} = -1.$

$\therefore$ AD_1, BE_2, CF_3 are concurrent (converse of Ceva's Th.).

$\therefore$ A, P, D_1 are collinear.

Again $\dfrac{BD_1}{CD_1} \cdot \dfrac{CE_1}{AE_1} \cdot \dfrac{AF_1}{BF_1} = -\dfrac{s-c}{s-b} \cdot \dfrac{s-b}{s} \cdot \dfrac{s}{s-c} = -1.$

$\therefore$ AD_1, BE_1, CF_1 are concurrent.

$\therefore$ A, D_1, Q are collinear.

$\therefore$ A, P, D_1, Q are collinear.

X, E_2, F_3 are collinear.

$\therefore$ $\dfrac{BX}{CX} \cdot \dfrac{CE_2}{AE_2} \cdot \dfrac{AF_3}{BF_3} = 1.$ (Menelaus' Theorem.)

$\therefore \frac{BX}{CX} = \frac{AE_2}{CE_2} \cdot \frac{BF_3}{AF_3} = \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = \frac{s-c}{s-b}$.

Similarly $\frac{CY}{AY} = \frac{s-a}{s-c}$ and $\frac{AZ}{BZ} = \frac{s-b}{s-a}$.

$\therefore \frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$.

$\therefore$ X, Y, Z are collinear (converse of Menelaus' Theorem).

608. AB, CF, DE are parallel and therefore pass through one point at infinity.

$\therefore$ the intersections of corresponding sides of $\triangle$s BCD, AFE are collinear, by Desargues' Theorem (Theorem 61).

I.e. P, Q, R are collinear.

Or thus: Let PQ cut AE, BD at R_1, R_2.

From $\triangle FPQ$, $\frac{FA}{PA} \cdot \frac{QE}{FE} \cdot \frac{PR_1}{QR_1} = 1$. (Menelaus.)

From $\triangle CPQ$, $\frac{CB}{PB} \cdot \frac{QD}{CD} \cdot \frac{PR_2}{QR_2} = 1$. (Menelaus.)

But, by parallels, $\frac{FA}{PA} = \frac{CB}{PB}$ and $\frac{QE}{FE} = \frac{QD}{CD}$,

$\therefore \frac{PR_1}{QR_1} = \frac{PR_2}{QR_2}$,

$\therefore$ R_1 and R_2 coincide,

$\therefore$ etc.

609. Suppose O to be on the arc AB and L on the arc AC.

Let OX, OY, OZ be the perpendiculars on BC, CA, AB, and let XY cut OL at P.

Then $\angle YPL = \angle PXC = \angle BOZ$.

And $\angle ALP = \angle OBZ$.

But $\angle BOZ + \angle OBZ = 90°$.

$\therefore$ LA is $\perp$ to XY.

610. Let P, E, F be the mid-points of AA′, AC, AB.

Then $PE \cdot PF = \frac{1}{2}A'C \cdot \frac{1}{2}A'B = \frac{1}{4}A'A^2 = PA^2$.

But E, F are on the nine-points circle of $\triangle ABC$.

$\therefore$ P is on the rad. axis of the circumcircle and nine-points circle.

Similarly for the mid-points of BB′ and CC′.

611. A, B, C are the excentres of $\triangle$ DEF; hence this Ex. becomes the same as Ex. 130.

612. Let AD cut the $\perp$ to AB through B in X.

Then $\angle$ BXD $= 90^\circ - \frac{1}{2}$A $= \angle$ BDX; and BG is $\perp$ to DX,

$\therefore$ XG = GD.

Now AF = FX,

$\therefore$ AD = AF + FD = FX + FD = 2FG.

Let AB be the given hypotenuse, S its mid-point, and YZ the length for the bisector.

Through B, S describe any circle; in this circle place a chord PQ of length $= \frac{1}{2}$YZ which passes through A.

Turn this chord APQ about A to the position AFG where F is on the $\perp$ bisector of AB.

Then AF . AG = AP . AQ = AS . AB.

$\therefore$ B, S, F, G are concyclic; but $\angle$ BSF is a rt. $\angle$.

$\therefore \angle$ BGF is a rt. $\angle$.

$\therefore$ G lies on the circle on AB as diameter.

Draw AC so that $\angle$ GAC $= \angle$ BAG.

$\therefore$ etc., etc.

613. $\angle$ AOB : $\angle$ BOC : $\angle$ COA = 3 : 4 : 5.

$\therefore$ the angles are 90°, 120°, 150°.

$\angle$ AOB $= 90^\circ = \angle$ ADB. $\therefore$ A, O, D, B are concyclic.

$\therefore \angle$ OAD $= \angle$ OBC, and $\angle$ AOD $= 180^\circ - \angle$ ABC $= 120^\circ = \angle$ COB.

$\therefore$ $\triangle$s OCB, ODA are similar.

Again, in $\triangle$s OCD, OBA,

$\angle$ COD $= \angle$ BOC $- \angle$ DOB $= 120^\circ - \angle$ DAB $= 90^\circ = \angle$ BOA,

and $\angle$ ODC $= \angle$ OAB ($\because$ A, O, D, B are concyclic).

$\therefore$ $\triangle$s OCD, OBA are similar, and $\triangle$OAC is common to $\triangle$s ABC, ADC.

614. (i) *If* R *is outside both circles or inside both.*

$\angle$ DPR $= \angle$ DBC and $\angle$ DQR $= \angle$ ABD,

$\therefore \angle$ DPR + $\angle$ DQR $= 180^\circ$. $\therefore$ D, P, R, Q are concyclic.

(ii) *If* R *is inside one circle, say* ABD, *and outside the other.*

$\angle DPR = 180° - \angle DPC = 180° - \angle DBC = \angle ABD$

$= 180° - \angle RQD.$

$\therefore$ D, P, R, Q are concyclic.

615. Let C, C′ be the centres of the given circles, P any point on the circle on SS′ as diameter, and PT, PT′ tangents drawn to the given circles.

Then $\odot PSS'$ is an Apollonius circle with respect to C, C′.

$\therefore PC : PC' = CS : C'S = CT : C'T'.$

$\therefore \triangle s\ PCT, PC'T'$ are similar.

$\therefore$ the circles subtend equal angles at P.

616. Let C, D be the centres of S, S′.

Let the common tangent touch S, S′ at P, Q and cut CD at T; draw $PN \perp$ to CT.

CP is $\parallel$ to DQ and CD = CP.

$\therefore \angle QDP = \angle DPC = \angle PDC.$

Hence $\triangle DPQ \equiv \triangle DPN.$

$\therefore$ N is on the circle S′.

$\therefore$ etc.

617. See Ex. 601.

618. Draw EP a tangent to the circle ABCD.

Since A, B, C, D are concyclic and FG is $\parallel$ to CD,

$\therefore$ A, B, F, G are concyclic.

$\therefore EP^2 = EB \cdot EA = EF \cdot EG.$

$\therefore \odot PFG$ touches $\odot ABCD$ at P.

619. $A_2 = \dfrac{B_1 + C_1}{2} = \dfrac{\pi}{2} - \dfrac{A_1}{2}.$

$\therefore A_3 = \dfrac{\pi}{2} - \dfrac{A_2}{2} = \dfrac{\pi}{2} - \dfrac{1}{2}\left(\dfrac{\pi}{2} - \dfrac{A_1}{2}\right) = \dfrac{\pi}{2} - \dfrac{\pi}{2^2} + \dfrac{A_1}{2^2}.$

Similarly $A_4 = \dfrac{\pi}{2} - \dfrac{\pi}{2^2} + \dfrac{\pi}{2^3} - \dfrac{A_1}{2^3},$

and $A_n = \dfrac{\pi}{2} - \dfrac{\pi}{2^2} + \dfrac{\pi}{2^3} \ldots + (-1)^n \dfrac{\pi}{2^{n-1}} + (-1)^{n-1} \dfrac{A_1}{2^{n-1}}$

$$= \frac{\pi}{2}\left(1 - \frac{1}{2} + \frac{1}{2^2} - \ldots \text{ to } \overline{n-1} \text{ terms}\right) + (-1)^{n-1}\frac{A_1}{2^{n-1}}$$

$$= \frac{\pi}{2}\,\frac{1-(-\frac{1}{2})^{n-1}}{1+\frac{1}{2}} + (-1)^{n-1}\frac{A_1}{2^{n-1}}$$

$$= \frac{\pi}{3}\left[1-(-\tfrac{1}{2})^{n-1}\right] + (-\tfrac{1}{2})^{n-1}A_1.$$

$\therefore A_\infty = \frac{\pi}{3}$; similarly $B_\infty = C_\infty = \frac{\pi}{3}$.

The base B_1C_1 has to rotate through an angle $\pi - \frac{C_1}{2} + \frac{B_1}{2}$

to get to the position B_2C_2.

The base B_2C_2 has to rotate through an angle $\pi - \frac{C_2}{2} + \frac{B_2}{2}$

to get to the position B_3C_3.

$\therefore$ the base B_1C_1 has to rotate through an angle

$$\sum_{r=1}^{r=n-1}\left(\pi - \frac{C_r}{2} + \frac{B_r}{2}\right)$$

to get to the position B_nC_n.

$\therefore$ the angle between B_1C_1, $B_\infty C_\infty$

$$= \tfrac{1}{2}\sum_{r=1}^{r=\infty}(B_r - C_r) = \tfrac{1}{3}(B_1 - C_1).$$

Now, if H, S are the orthocentre and circumcentre of $\triangle A_1B_1C_1$, then $\angle SA_1H = (B_1 - C_1)$.

Let $\angle HA_1X = \frac{1}{3}(B_1 - C_1)$.

Then $B_\infty C_\infty$ is $\perp$ to A_1X.

620. Take any point A′ on WX, draw A′B′ ∥ to AB to cut XY at B′, and draw B′C′, C′D′ ∥ to BC, CD; join A′D′.

Then $\frac{AA'}{DD'} = \frac{AA'}{BB'}\cdot\frac{BB'}{CC'}\cdot\frac{CC'}{DD'} = \frac{XA}{XB}\cdot\frac{YB}{YC}\cdot\frac{ZC}{ZD}$

$$= \frac{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)}\cdot\frac{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)}\cdot\frac{\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{D}{2}\right)} = \frac{WA}{WD}.$$

$\therefore$ A′D′ is $\parallel$ to AD.

Project each side of ABCD on the corresponding side of A′B′C′D′; then the perimeter of A′B′C′D′ = that of ABCD together with certain projections (to be taken with regard to sense).

Now $\angle$ A′AD = $\angle$ AA′B′.

$\therefore$ the projection of AA′ on AD = the projection of AA′ on A′B′.

$\therefore$ the projection at A to be added to (or subtracted from) A′B′ = that to be subtracted from (or added to) A′D′.

Similarly at B, C, D.

$\therefore$ the perimeters of ABCD, A′B′C′D′ are equal.

Again A′ was chosen arbitrarily.

$\therefore$ there are an infinite number of positions for A′B′C′D′.

621. Describe a circle on BC as diameter. On AB describe an arc to contain an angle of 45°. Let them intersect at X. Draw AY, AZ $\perp$ to XB, XC. Then AYXZ is the required square.

622. $\triangle$ BMP : $\triangle$ PLM = $\triangle$ BMP : $\triangle$ AMP = BM : MA = BM : PL.

Similarly $\triangle$ PLM : $\triangle$ PLC = AL : LC = MP : LC.

Now $\triangle$s BMP, PLC are similar. $\therefore$ BM : PL = MP : LC.

$\therefore$ $\triangle$ BMP : $\triangle$ PLM = $\triangle$ PLM : $\triangle$ PLC.

623. Draw OL $\parallel$ to PQ to cut AB in L; let XY cut OA in M, and OL in S. We will shew that S is a fixed point.

As $\angle$ AOL = $\angle$ APX = $\angle$ B (or $180° - \angle$ B), OL and L are fixed.

Considering $\triangle$OAL, and line XYS

$$\frac{LS}{OS} \cdot \frac{OM}{AM} \cdot \frac{AX}{LX} = 1,$$

$$\therefore \frac{LS}{OS} = \frac{AM}{OM} \cdot \frac{LX}{AX} = \frac{AM}{OM} \cdot \frac{OP}{AP}.$$

Now, by property of complete quadrilateral X {AP, YO} = − 1.

$\therefore$ {AP, MO} = − 1, $\therefore$ $\frac{AM \,.\, OP}{AP \,.\, OM} = \frac{1}{2}$. (See p. 125.)

$\therefore$ LS = $\frac{1}{2}$OS = OL, and S is fixed.

Y is intersection of AQ, BP; and there is a (1, 1) correspondence between the positions of AQ, BP. Therefore the locus of Y is a conic through A, B; and, taking particular case, the locus passes through O. It also passes through the orthocentre of $\triangle$OAB, as is seen by making AB the diameter of the $\odot$, $\therefore$ the locus of Y is a rectangular hyperbola.

624. Let A be the given circle, and B, C, ... circles of the coaxal system.

Let the rad. axis of A, B cut the rad. axis of the system at P.

Then P is on the rad. axis of A, B and on that of B, C.

$\therefore$ P is on the rad. axis of A, C.

Similarly P is on the rad. axis of A and any circle of the system.

(i) If a circle is described which passes through a fixed point and which is coaxal with a given circle and one of a system of concentric circles, then all the circles which can be so described pass through another fixed point.

(ii) If a circle is described which passes through a fixed point and which is such that its rad. axis with a fixed circle passes through a fixed point, then all the circles which can be so described pass through another fixed point.

(iii) The same as (i) with "a system of coaxal circles" substituted for "a system of concentric circles."

625. Let AD, BC intersect at X; let E, F be the mid-points of AC, BD. Then EF is $\parallel$ to DC, AB.

$\therefore$ the common chord of the circles on AC, BD as diameters is $\perp$ to the join of their centres, i.e. $\perp$ to EF, $\therefore$ to AB, CD.

Draw BN, AM $\perp$ to DX, CX.

Then the BD ⊙ passes through N, and the AC ⊙ through M.
Now X is on the common chord of the circles, if

$$XN \cdot XD = XM \cdot XC.$$

We have $XN \cdot XA = XM \cdot XB,$

and $XD : XA = XB : XC.$

$\therefore\ XM \cdot XD = XM \cdot XC.$

$\therefore$ X is on the common chord of the two circles.

626. Let PC cut AB in X; and on AB take Y so that

$$AY : BY = -AX : BX.$$

Let PY cut the ⊙ in D; then $\{AB, XY\} = -1$;

$\therefore\ P(AB, CD) = -1.$

Let AB, CD meet in M; AC, BD in N; BC, AD in O.

Then O is the pole of MN; i.e. the pole of MN lies on BC.

$\therefore$ the pole of BC lies on MN, etc.

$\therefore$ the tangents at B, C intersect on MN.

Similarly the tangents at A, D intersect on MN.

627. See Ex. 433.

All the circles pass through two fixed points on the line of centres of the given circles.

$\therefore$ the locus of the centre is a straight line $\perp$ to that line.

628. Let BC, CA, AB meet the corresponding ⊙s in P_1, P_2; Q_1, Q_2; R_1, R_2. Let X, Y, Z be the mid-points of P_1P_2, Q_1Q_2, R_1R_2. Since A is on the radical axis of the Q and R ⊙s, $AQ_1 \cdot AQ_2 = AR_1 \cdot AR_2$. $\therefore\ AY^2 - YQ^2 = AZ^2 - ZR^2$. Similarly $BZ^2 - ZR^2 = BX^2 - XP^2$ and $CX^2 - XP^2 = CY^2 - YQ^2$. $\therefore\ AY^2 + BZ^2 + CX^2 = AZ^2 + BX^2 + CY^2$. $\therefore$ the $\perp$ bisectors of P_1P_2, Q_1Q_2, R_1R_2 are concurrent (Ex. 129). Also $Q_1Q_2R_1R_2$ are concyclic, as also $R_1R_2P_1P_2$: and these two ⊙s, having the same centre, are identical, $\therefore$ $P_1P_2Q_1Q_2R_1R_2$ are concyclic.

Let OP_1 meet its circle again in p_1; etc. Invert from O, taking the tangent from O as radius of inversion. Then the three ⊙s invert into themselves; P_1 to p_1, etc.;

$$\odot P_1P_2Q_1Q_2R_1R_2 \text{ becomes } \odot p_1p_2q_1q_2r_1r_2.$$

If q_1q_2 meets r_1r_2 in L; then $\angle q_1 \cdot \angle q_2 = \angle r_1 \cdot \angle r_2$, $\therefore$ L is on OA; etc.

629. Draw OD $\perp$ to EF, to meet AB in D. Then, since OC bisects EF, and OD is $\parallel$ to EF, $O\{EF, CD\} = -1$.

$\therefore$ $\{AB, CD\} = -1$, $\therefore$ D is a fixed point.

Again $\angle DOF = \angle OFE = \angle EBA$, $\therefore$ DO touches $\odot OAB$ at O.

$\therefore$ $DO^2 = DB \cdot DC = \text{const.}$

$\therefore$ locus of O is $\odot$ with D as centre, cutting original $\odot$ orthogonally.

630. (i) Take a pair of rectangular axes through O.

Let P be $(\bar{x}, \bar{y})$ and A, B, C, ... $(x_1, y_1)(x_2, y_2)(x_3, y_3)$ ....

Then
$$\begin{aligned}\Sigma AP^2 &= \Sigma (x_1 - \bar{x})^2 + \Sigma (y_1 - \bar{y})^2 \\ &= \Sigma (x_1^2 + y_1^2) + n \Sigma (\bar{x}^2 + \bar{y}^2) - 2\bar{x}\Sigma x_1 - 2\bar{y}\Sigma y_1 \\ &= \Sigma OA^2 + n \cdot OP^2 \quad (\Sigma x_1 = 0,\ \Sigma y_1 = 0, \text{ for O is centroid of A, B, C, ...}) \\ &= n \cdot OP^2 + \text{const.}\end{aligned}$$

(ii) Let the feet of the $\perp$s from P upon OA, OB, ... be N_1, N_2, ...

Then for a given position of P all the points N lie on a $\odot$ whose diameter is OP.

And as the angles between successive positions of ON are equal, the N points are distributed evenly round the circumference. $\therefore$ their centroid is the centre of the $\odot$, i.e. the mid-point of OP. $\therefore$ as P moves round its $\odot$, the centroid moves round a concentric $\odot$ of half the radius.

631. (i) Let the distances of A, B, C, P, measured from any origin on ABC be a, b, c, x; regard being had to sign. Then it must be shewn that

$$\Sigma (c - b)(x - a)^2 \equiv (b - c)(c - a)(a - b).$$

Now
$$\begin{aligned}\Sigma (c - b)(x - a)^2 &\equiv x^2 \Sigma (c - b) - 2x\Sigma a (c - b) + \Sigma a^2 (c - b) \\ &\equiv 0 - 0 + (b - c)(c - a)(a - b).\end{aligned}$$

(ii) Suppose that the points are in order A, B, C. Let L, M, N be the mid-points of AB, AC, BC; O the centre of the ⊙ that touches the ⊙ on AB, AC, BC. Let AB $= 2p$, BC $= 2q$; then LM $= q$, MN $= p$. Let r be the radius of the touching ⊙. Then OL $= r + p$, ON $= r + q$, OM $= p + q - r$. Let h be the projection of OM on LN.

Then from △ OLM, $(r+p)^2 = (p+q-r)^2 + q^2 \mp 2hq$ (according to angle)

and from △ OMN, $(r+q)^2 = (p+q-r)^2 + p^2 \pm 2hp$,

$\therefore\ p(r+p)^2 + q(r+q)^2 = (p+q)(p+q-r)^2 + pq(p+q)$,

$\therefore\ 2r(p^2+q^2+\overline{p+q}^2) = (p+q)^3 - p^3 - q^3 + pq(p+q)$
$= 4pq(p+q)$.

$\therefore\ r = \dfrac{pq(p+q)}{p^2+q^2+pq}$.

632. Fixing our attention on the triangle formed by the tangents, we have to show that the chords of contact of the in-circle meet the opposite sides in collinear points.
See Ex. 195.

633. (i) Let O be the centre of any circle through L, L'; then O is on the rad. axis.
$\therefore$ the length of the tangent from O to any circle of the system is OL.
$\therefore$ the circle with centre O cuts the circles of the system orthogonally.

(ii) See Ex. 417.

634. *Lemma.* If $S = 0$, $S' = 0$ be two circles, in the form $x^2 + y^2 + 2gx + 2fy + c = 0$, then S, S' are the squares of the tangents to the ⊙s from (x, y). Also $S = \lambda S'$ is a coaxal circle. The form of this equation shews that "the ratio of the tangents drawn to two circles from any point on a third coaxal ⊙ is constant." (Also see proof of Ex. 436.)

In the present case L is one of the coaxal ⊙s,

$\therefore$ PA : PL = QA : QL = RB : RL = SB : SL.

Taking the figure in which ⊙ PQRS contains the other ⊙ and L, and Q, R are on the same side of AB, we see that LA bisects $\angle$ PLQ and LB bisects $\angle$ RLS: and all these half angles are acute. Also, in $\triangle$s ALQ, BLR, $\angle$ A = $\angle$ B and QA : QL = RB : RL, $\therefore$ $\angle$ ALQ = $\angle$ BLR;
similarly $\angle$ ALP = $\angle$ BLS, $\therefore$ $\angle$ PLR = $\angle$ QLS.

635. See Ex. 290.

636. $\because$ CP is a diameter of the circle through C, L,
$\therefore$ $\angle$ CLP is a rt. $\angle$.
$\therefore$ PL is the polar of L′ with regard to the circle with centre C′. (Ex. 415.)
$\therefore$ the polar of P passes through L′.

637. Let N be the mid-point of the chord PQ of ⊙ whose centre is O, radius r; let the other ⊙ have centre C, radius R:
$CP^2 - R^2 + CQ^2 - R^2 = \text{const.}$
$\therefore$ $CP^2 + CQ^2 = \text{const.}$ $\therefore$ $CN^2 + NP^2 = \text{const.}$
$\therefore$ $CN^2 + r^2 - ON^2 = \text{const.}$ $\therefore$ $CN^2 - ON^2 = \text{const.}$
$\therefore$ locus of N is straight line $\perp$ to OC.

638. (i) See Ex. 301.

(ii) Remember that, if X be a centre of similitude, and XRR′ cut the ⊙s in R and R′, then R, R′ are either corresponding points with regard to similitude; or else inverse points when the one ⊙ is inverted into the other. The only case in which they are both is that in which XRR′ is a common tangent.

Let O, C, C′ be centres of the 3 ⊙s; let QP, QP′ cut the corresponding ⊙s in R, R′.

Then $\angle$ OQP = $\angle$ OPQ = $\angle$ CPR = $\angle$ CRP.
$\therefore$ CR is $\parallel$ to QO; so also C′R′ is $\parallel$ to QO.
$\therefore$ C′R′ is $\parallel$ to CR and R, R′ are corresponding points; and XRR′ is a straight line.

Now QP . QR = QP′ . QR′, ∴ P, P′, R′, R, are concyclic; and XR . XR′ = XP . XP′, ∴ R, R′ are inverse points. As they are also corresponding, they are the points of contact of a common tangent, to which OQ is ⊥ and the tangent at Q is ∥.

639. (i) *Properties which persist after inversion.*

(*a*) Number of points common to two loci.

(*b*) Coincidence of points; i.e. contact of various orders.

(*c*) Angles of intersection of loci.

(*d*) In general, circles become circles.

(*e*) Inverse points with regard to a circle become inverse with regard to inverse circle.

(ii) Since circles B and C cut orthogonally,
∴ ⊙B, when inverted with respect to ⊙ C, inverts into itself.
Now Q, Q′ are inverse points with respect to ⊙ B.
If we invert with respect to ⊙ C, by Ex. 482, R, R′ (the inverses of Q, Q′) will be inverse points with respect to ⊙B (which is the inverse of ⊙B).

640. Let O, C be the centres of ⊙s PAB, PRQ.
∠s APB, AQB are constant, being subtended by a fixed chord of their respective ⊙s.
∴ ∠ QBR is constant, ∴ QR is const.
Now △s APB, RPQ are similar.
∴ their circumradii are proportional to the bases of the triangles; i.e. is a fixed ratio.
∴ circumradius of △PQR is constant.
Furthermore, ∠ CPQ = ∠ OPB, ∴ PC is ⊥ to AB and in a fixed direction.
∴ locus of C is a ⊙ equal to ⊙APB, and moved ∥ to line of centres through a distance PC.

641. $\triangle$s ABB′, ADD′ are congruent, having AB = AD, AB′ = AD′, and $\angle$ BAB′ = $\angle$ DAD′. Corresponding lines in the two $\triangle$s are at rt. $\angle$ s.

$\therefore$ BB′, DD′ are equidistant from A, and at rt. $\angle$ s.

These are therefore $\perp$ tangents to a $\odot$ of centre A.

The external bisector of the angle between these tangents is $\sqrt{2}$ times as far from A as either of the tangents.

Now, comparing $\triangle$s ABB′, ACC′, AC : AB = AC′ : AB′ $\sqrt{2}$: 1; and $\angle$ BAB′ = $\angle$ CAC′. $\therefore$ the $\triangle$s are similar; CC′ makes $\angle$ 45° with BB′ and DD′; and the ratio of the distances from A of CC′, BB′ is $\sqrt{2}$: 1.

$\therefore$ CC′ is identical with the bisector, not passing through A, of the angle between BB′ and DD′.

642. Move A and B $\parallel$ to PQ through distances equal to CC′, and let A, B take up positions X, Y. The problem now is to find C′ on PQ, so that $\triangle$ C′XY = $\triangle$ C′A′B′.

If P be such a point in the plane that $\triangle$ PXY = $\triangle$ PA′B′, the ratio of the distances of P from XY and A′B′ is fixed. The locus of P is therefore a pair of straight lines through the intersection of XY and A′B′ (a pair of lines, since the signs of the $\perp$ s are immaterial).

Construct these lines; then the points where they meet PQ are the positions of C′.

643. (i) *If the sides of* A *are even in number* $(2n)$, join alternate vertices. The result will be a polygon C of n sides, the sides of C being double and parallel to alternate sides of B.

If polygon C be moved $\parallel$ to itself to any other position C′ and the vertices of C′ be joined to vertices of B and produced, so as to form a polygon A′ as in the original arrangement, it will be obvious that the sides of A′ are bisected by the vertices of B.

(ii) If the sides of A are odd in number $(2n + 1)$, n sides of a polygon C may be formed as in (i), but the

$(n+1)$th side will be identical with a side of A, say with XY, passing through a vertex, P, of B.

If an attempt be made to construct a second figure of the same properties, by shifting a side of C′ ∥ to itself, and building up a second polygon A′, it will be found that all will go well till the side X′Y′ is reached. But X′Y′ will be ∥ to XY, and therefore cannot pass through P.

644. Let a be the mid-point of BC; and draw AL ∥ to BC, meeting QR in L. Let Aa meet QR in X.

A $\{BC, aL\} = -1$, ∴ the pole of AL with respect to the ⊙ lies on AX.

Again the pole of QR lies on AL, ∴ the pole of AL lies on QR. ∴ X is the pole of AL, ∴ the ⊥ from X to AL is a diameter and passes through P.

645. Let O be the centre, r the radius, of any coaxal ⊙.

The ⊙ on PQ as diameter is orthogonal to the O circle.

Let OQ cut this orthogonal ⊙ in N.

Then ∠ PNQ = 90°, and ON . OQ = (tangent to PQ⊙)$^2 = r^2$.

∴ NP is polar of Q with regard to O circle.

646. Let O be the centre of the ⊙; and let S be the circum-circle of the square PQP′Q′. Let OP, OQ meet S in M, N respectively.

Then, since PP′, QQ′ are diameters of S, P′M is ⊥ to OP; and Q′N is ⊥ to OQ. Now OP . OM = OQ . ON. But if P, P′ are conjugate, P′M is the polar of P, then OP . OM = (rad. of O circle)^2.

∴ OQ . ON = (radius)^2, ∴ Q′N is polar of Q, ∴ Q, Q′ are conjugate.

647. Let L be the given line; O the interior centre of similitude of the ⊙s.

Through O draw a line ∥ to L, cutting the ⊙s in PP′, QQ′ respectively; the order of these points being P, P′, Q′, Q.

Then PQ is the longest and P′Q′ the shortest straight line from the one ⊙ to the other, ∥ to L.

Draw the tangents at P, P′, Q′, Q; let any other straight line ∥ to L cut these tangents in p, p', q', q; and the ⊙s in a, $b : c$, d. The order of these points is $pabp'q'cdq$.

Now Pp is ∥ to Qq, ∴ PQ = $pq > ad$;

and P′p′ is ∥ to Q′q′, ∴ P′Q′ = $p'q' < bc$.

648. Invert with regard to O, and let the inverse points be denoted by small letters. We have chords pp', qq' of a fixed ⊙ meeting at a fixed point O; pq, $p'q'$ meet at s; what is the locus of s?

Let $p'q$, pq' meet at t; then st is the polar of O (Th. 50), and ∴ a fixed line.

The locus required in the original problem is therefore a circle through O.

649. Let YU, ZW meet AC in F, F′. Let YQ, ZR meet AD in S, P.

Then $\{AFQR\} = Y\{AFQR\} = \{AUSV\} = Q\{AUSV\} = \{AXYZ\}$
$= R\{AXYZ\} = \{AWVP\} = Z\{AWVP\} = \{AF'QR\}$.

∴ F coincides with F′.

650. △s PAB, QED are such that the intersections of three pairs of sides are collinear; viz. AB, ED; PA, QE; PB, QD. ∴ the triangles are in perspective; and PQ, AE, BD are concurrent.

651. Let AC, BD meet in X; AD, CB in Y; AB, CD in Z; AB, XY in T.

First, Z is a fixed point. For if any other ⊙ be drawn through AB, to cut the fixed ⊙ in C′, D′; then AB, CD, C′D′ are concurrent.

Secondly, $Y\{AB, XZ\} = -1$; ∴ $\{AB, TZ\} = -1$; ∴ T is a fixed point.

652. Draw AL, BM ∥ to A′B′C′.

Then $A'\{CBAC'\} = A\{C'B'A'L\}$ by ∥s

$= B\{C'B'A'M\}$

$= B'\{CBAC'\}$ by ∥s.

∴ CBA are collinear.

653. Invert with regard to A and let B′, P′, Q′ be the inverses of B, P, Q.

Draw P′H, Q′K tangents to ⊙s AB′P′, AB′Q′.

Then $\angle APB = \angle HP'A - \angle P'B'A$, similarly $\angle AQB = \angle Q'B'A$,

$\therefore \angle P'B'Q' = \angle P'B'A \sim \angle Q'B'A = \angle APB \sim \angle AQB = \text{const.}$

Again, if k is the radius of inversion,

$$AP = \frac{k^2}{AP'}; \quad AQ = \frac{k^2}{AQ'}; \quad PB = \frac{k^2}{AP' \cdot AB'} \cdot P'B';$$

$$QB = \frac{k^2}{AQ' \cdot AB'} \cdot Q'B'.$$

$$\therefore \frac{AP}{PB} : \frac{AQ}{QB} = \frac{Q'B'}{P'B'} : 1,$$

$$\therefore \frac{Q'B'}{P'B'} \text{ is constant.}$$

Then, since $\frac{Q'B'}{P'B'}$ is constant and $\angle P'B'Q'$ is constant, Q′ describes a locus similar to the locus of P′. (Chap. VIII. § 3.)

But P′ is the inverse of P, and P describes a circle. ∴ P′ describes a circle or a straight line.

∴ Q′ describes a circle or a straight line.

∴ Q describes a circle or a straight line.

654. Draw CL ∥ to DE and BA; BN ∥ to DF and CA.

Let AXY meet BC in M.

$$C\{DXEL\} = A\{DXEB\} = \{DMCB\}.$$

Similarly $B\{DYNF\} = A\{DYCF\} = \{DMCB\}$.

∴ $C\{DXEL\} = B\{DYNF\}$. But CD, CE, CL are ⊥ to BD, BN, BF respectively. ∴ CX is ∥ to BY.

655. (i) ABC is an Apollonius ⊙ with regard to the fixed points P, Q. ∴ PQ is divided harmonically by the ⊙, of which it is a diameter. ∴ P, Q are inverse points.

(ii) Let the diameter bisecting BC at rt. ∠s be XOY: let AB, AC meet this diameter in L, M.

Then AX, AY are the bisectors of ∠ BAC; and

$$A\{BC, XY\} = -1, \quad \therefore \{LM, XY\} = -1,$$

∴ L, M are inverse points.

656. This is really a particular case of the theorem referred to in Ex. 634, that if S, S′ be two circles, the ratio of the tangents drawn to these ⊙s from any point of a coaxal ⊙ is constant. For S, and the point P are two ⊙s; and ABC is a coaxal ⊙.

Otherwise, as follows. Let O, O′ be centres of ⊙s S, ABC. Draw AN ⊥ to OO′.

Then (tangent from A)²

$$= AO^2 - OP^2 = AO'^2 + OO'^2 - 2O'O \cdot O'N - OP^2$$
$$= O'P^2 + OO'^2 - 2OO' \cdot O'N - OP^2$$
$$= OO'^2 + (O'P - OP)(O'P + OP) - 2OO' \cdot O'N$$
$$= OO'(OO' - O'P - OP) + 2OO' \cdot O'N$$
$$= -2OO' \cdot O'P + 2OO' \cdot O'N = 2OO' \cdot NP,$$

and $$AP^2 = AO'^2 + O'P^2 - 2O'P \cdot O'N$$
$$= 2O'P \cdot NP.$$

∴ tangent to S : distance from P is constant for all points on ⊙ABC.

If S becomes ∝, the (tangent)² becomes infinite, but always varies as PN. We therefore have that the distance of a point on a ⊙ from the tangent to that ⊙ at P varies as the (distance from P)².

657. Let PQ, RS intersect at X. Then $Xp \cdot XQ = XR \cdot XS$; $Xq \cdot XP = XR \cdot XS$; $Xr \cdot XS = XP \cdot XQ$; $Xs \cdot XR = XP \cdot XQ$

Also XP = XR, XQ = XS. Hence $Xp = Xr = XP = XR$; and X is centre of ⊙ *pr*PR.

Similarly X is centre of ⊙ QS*qs*.

Let A, B be centres of ⊙s RP, QS. Then ∠ AXR = ∠ XBS; and ∠s R, S are rt. ∠s. ∴ XR : AR = BS : XS.

∴ XR . XS = AR . BS.

658. Let ABC be the fixed △; PQR the inscribed △ *given in specie*. Let ⊙s BPR, CPQ meet at O; then

∠ QOR = 360° − ∠ POR − ∠ POQ = B + C = 180° − A.

∴ ⊙AQR passes through O. ∴ QR, RP, PQ subtend fixed ∠s at O, and figure PQRO is given in specie, ∴ ∠s QRO, QPO are fixed, ∴ ∠s QAO, QCO are fixed, and O is a fixed point. Accordingly PQR revolves about O, varying its size; any point G which moves so as to form given shape with PQR describes a locus similar to that of P (or Q, or R), i.e. a straight line. G may be centroid, or orthocentre, or any such point.

Since given angles P, Q, R may be fitted on to sides BC, CA, AB in 6 different ways, the locus is generally 6 straight lines. If G happens to coincide with O, G will not move.

659. Let P, C be the centres of ⊙s U, V.

The tangent from O to U and that from O to V are of fixed lengths.

∴ the difference of their squares is constant.

∴ 2OY . CP = constant (Ex. 393),

and OY is ∥ to CP.

∴ Y describes a locus similar to the inverse of the O circle with regard to C.

∴ the locus of Y is a circle (or a straight line if C is on the O circle).

660. Let the centres be A, B; the radii a, b; the perpendiculars from the centres p, q. Let the chord cut the ⊙s in PQ, RS; M, N being mid-points of PQ, RS. Draw AY ⊥ to BN.

Then $PM^2 = MR \cdot MS = MN^2 - RN^2$.

$\therefore\ a^2 - p^2 = AY^2 - b^2 + q^2$

$= AB^2 - (q-p)^2 - b^2 + q^2$.

$\therefore\ 2pq = a^2 + b^2 - AB^2$.

661. Let the mid-points of OP, OQ, OR be p, q, r; AO, BO, CO meet B′C′, C′A′, A′B′ respectively in L, M, N; A′P′, B′Q′, C′R′ meet B′C′, C′A′, A′B′ in F, G, H.

Then $\{B'C', LF\} = \{rq, OP'\} = \dfrac{rO}{Oq} \cdot \dfrac{P'q}{rP'} = \dfrac{rO}{Oq} \cdot \dfrac{rO}{Oq} = \dfrac{Or^2}{Oq^2}$,

$\therefore\ \Pi\{B'C', LF\} = 1$,

$\therefore\ \Pi \dfrac{B'L}{LC'} = \Pi \dfrac{B'F}{FC'}$.

But $\Pi \dfrac{B'L}{LC'} = -1$, since A′L, B′M, C′N concur,

$\therefore\ \Pi \dfrac{B'F}{FC'} = -1$; $\therefore$ A′P′, B′Q′, C′R′ concur.

662. AA′BB′ are concyclic, $\therefore\ \angle OA'B' = \angle OBA$.

Similarly $\angle OA'D' = \angle ODA$.

$\therefore\ \angle B'A'D' = \angle OBA$ and $\angle ODA = 180^\circ - \angle BAD$,

and so for the other angles.

If A′B′C′D′ be turned over, hinging about one of the bisectors of the angles between AC and BD, its sides will be $\parallel$ to those of ABCD, and its diagonals will coincide with those of ABCD.

663. Let ABCD be the quadrilateral, and P any point on its circumcircle.

Draw PL, PM, PN, PR $\perp$ to AB, AC, BD, CD.

$PL = PA \sin PAB$, $PR = PD \sin PDC$, $PM = PA \sin PAC$,

$PN = PD \sin PDB$.

Now $\sin PAB = \sin PDB$, and $\sin PDC = \sin PAC$.

$\therefore\ PL \cdot PR = PM \cdot PN$.

664. Each circle of the system passes through two fixed points X, X′ say (proof as in Ex. 433).

∴ the system is a coaxal system of the intersecting type.

Let P be the given point.

Describe a circle through P coaxal with the pt.-circles X, X′.

Let PP′ be a diameter of this circle, and C the centre of one of the circles through X, X′.

Let CP cut ⊙ on PP′ as diameter at N.

Since ⊙ PNP′ cuts any ⊙ through X, X′ orthogonally, ∴ CN . CP = CX², and ∠ PNP′ is a rt. ∠.

∴ P′N is the polar of P with respect to the circle with centre C. But P′ is independent of the position of C.

∴ the polar of P with respect to any circle through X, X′ passes through a fixed point P′.

I.e. the polar of P with respect to any circle of the given system passes through a fixed point.

665. Let the line cut the one circle in A, B and the other circle in C, D.

The problem at once reduces to finding P, Q in the line such that P, Q are harmonic conjugates with regard to A, B and also with regard to C, D. See Ex. 243.

666. Through E, F draw EG, FH ∥ to DA, BA to meet these lines in G, H. Then △ AGD = △ AED = △ AFB = △ AHB.

∴ AD . AG sin BAD = AB . AH sin BAD. ∴ $\frac{AD}{AB} = \frac{AH}{AG}$ and GH is ∥ to BD. Again △s BEG, DHF are similar.

∴ $\frac{DF}{DH} = \frac{BE}{BG}$. But $\frac{DH}{AD} = \frac{BG}{AB}$.

∴ $\frac{DF}{AD} = \frac{BE}{AB}$; and since ∠ D = ∠ B, ∴ △s ADF, ABE are similar, and ∠ DAF = ∠ BAE.

Draw EM, FN ⊥ to AD, AB. Then EM . AD = FN . AB.

But ∠ DAE = ∠ BAF. ∴ $\frac{EM}{AM} = \frac{FN}{AN}$. ∴ AM . AD = AN . AB.

Now ⊙ on ED passes through M; and AM . AD = square of tangent to this ⊙ from A. So AN . AB = square of tangent to FB ⊙ from A.

∴ A is on radical axis of these two circles.

667. The ⊙ cuts OX in A, B; OY in C, D. AC, BD meet in fixed point P; let AD, BC meet in Q.

Then O {AD, PQ} = −1; ∴ OQ is fixed.

Again OQ is polar of P, ∴ centre lies on ⊥ from P to OQ, a fixed line.

668. Invert with regard to O. The inverse theorem is:

"If two circles intersect orthogonally at p, and a common tangent touches them at q, q', then the angle between pq, pq' is half a right angle."

If c, c' are the centres of the circles.

Then
$$\angle pqq' + \angle pq'q = \tfrac{1}{2}\angle pcq + \tfrac{1}{2}\angle pc'q'$$
$$= \tfrac{1}{2}(\angle pcc' + \angle pc'c)$$
$$= \tfrac{1}{2} \text{ a rt. } \angle. \quad (\because \angle cpc' \text{ is a rt. } \angle).$$

∴ the inverse theorem is true.

∴ the given theorem is true.

669. Let r be radius of given ⊙, and O its centre.

Since A_1C_1 is polar of B_2, and B_2C_2 of A_1, ∴ P_1 is pole of A_1B_2; similarly P_2 of A_2B_1. Let OP_1 meet A_1B_2 in M_1, and ⊙ $P_1A_1B_2$ in N_1; with similar notation for OP_2.

Since P_1 is pole of A_1B_2, ∴ $OP_1 . OM_1 = r^2$. Now O is orthocentre of $\triangle P_1A_1B_2$ (Ex. 564). ∴ $ON_1 = 2OM_1$. ∴ $OP_1 . ON_1 = 2r^2$. Similarly $OP_2 . ON_2 = 2r^2$.

∴ $OP_1 . ON_1 = OP_2 . ON_2$. ∴ O is on radical axis of ⊙s $P_1A_1B_2$ and $P_2A_2B_1$.

670. Let ABCD be the quadrilateral, EF its third diagonal, L, M, N the middle points of BD, AC, EF. Then LMN is a straight line (Ex. 593).

If A, C are conjugate with respect to a ⊙S; then ⊙ on AC is orthogonal to S; and conversely, the extremities

of a diameter of a ⊙ orthogonal to S are conjugate points with regard to S (Ex. 678).

Let AC, BD meet at G; AC, EF at H; BD, EF at K. $\{AC, GH\} = -1$. ∴ $MG \cdot MH = MC^2$. ∴ ⊙ on AC as diameter is orthogonal to ⊙ GHK. Similarly ⊙s on BD, EF are orthogonal to ⊙ GHK. As the centres of these three ⊙s are collinear, the ⊙s are coaxal. Any ⊙ S orthogonal to two of these ⊙s is orthogonal to the third. ∴ if two pairs of opposite vertices are conjugate with regard to S, so is the third pair.

All ⊙s S form a coaxal set having LMN as radical axis.

671. Let O_1, O_2, O_3 be the centres of C_1, C_2, C_3.

The three common chords pass through the radical centre, P say, of the three circles.

Now the common chord of C_2C_3 passes through O_1 and is ⊥ to O_2O_3.

And the common chord of C_3C_1 passes through O_2 and is ⊥ to O_3O_1.

∴ P is the orthocentre of △ $O_1O_2O_3$.

∴ the ⊥ to O_1O_2 through P passes through O_3.

But this ⊥ is the common chord of C_1C_2.

∴ the common chord of C_1C_2 passes through the centre of C_3.

672. Take any point R in the line l.

∵ the plane P touches all the spheres at O,

∴ the length of any tangent-line from R to any one of the spheres = RO.

∴ the tangent from R to any one of the circles in the plane Q = RO.

∴ the circles in the plane Q form a coaxal system of which L_1, L_2 are the limiting points, and l the rad. axis.

673. We lose non-generality by supposing that, if AB, CD be the segments that subtend equal angles, the order of

these points is A, B, C, D. For if AB, CD overlap, then AC, BD also subtend equal segments.

Let P be the variable point; and PE the internal bisector of $\angle BPC$ and $\angle APD$; PF the external bisector of the same angles; E, F being on ABCD.

Then $P\{EBAF\} \equiv P\{ECDF\}$, $\therefore \{EBAF\} = \{ECDF\}$.

$\therefore \dfrac{EB \cdot AF}{EF \cdot AB} = \dfrac{EC \cdot DF}{EF \cdot DC}$. $\therefore \dfrac{EB}{EC} \cdot \dfrac{AF}{DF} = \dfrac{AB}{DC}$.

Now $\{AD, EF\} = -1$; $\therefore \dfrac{AF}{DF} = -\dfrac{EA}{ED}$. $\therefore \dfrac{EB \cdot EA}{EC \cdot ED} = \dfrac{AB}{CD} = \text{const.}$

Again, $\dfrac{EB}{EC} = -\dfrac{FB}{FC}$. $\therefore \dfrac{FB \cdot FA}{FC \cdot FD} = \dfrac{AB}{CD} = \text{const.}$

This equation leads to a quadratic for the positions of E and F, which are thus shewn to be fixed points. The locus of P is a $\odot$ on diameter EF.

674. (i) If the fixed circles intersect, invert with regard to one of their common points.

The variable circles invert into two circles which touch one another and also touch each of a pair of straight lines.

Now the locus of the point of contact of these two circles is obviously the bisectors of the angles between the straight lines. $\therefore$ in the original figure the locus is one of two circles through the points of intersection of the fixed circles.

(ii) If the fixed circles do not intersect, invert with regard to one of the limiting points of the coaxal system determined by these two circles.

The variable circles invert into two circles which touch each other and also touch a pair of concentric circles.

Now the locus of the point of contact of the two variable circles is obviously one of two concentric circles.

$\therefore$ in the original figure the locus is one of two circles.

675. Invert with respect to the point of contact of A, B. These become ∥ straight lines. The three other points of contact become collinear, ∴ the original four points of contact are concyclic.

676. Each ⊙ becomes a straight line, each centre becomes the image of O in the line. The inverse theorem is therefore: "If AO is perp. to BC, and BO perp. to CA, then CO is perp. to AB."

677. Let AG meet OB in N. By Menelaus in △OLB, disregarding sign,

$$\frac{OM}{ML} \cdot \frac{LA}{AB} \cdot \frac{BN}{NO} = 1.$$

$$\therefore \frac{OL - OM}{OM} = \frac{LA}{AB} \cdot \frac{BN}{NO}.$$

Now, by similar △s,

$$\frac{BN}{NO} = \frac{BG}{AO} = \frac{OB}{AO}.$$

$$\therefore \frac{OL}{OM} = 1 + \frac{LA \,.\, OB}{AB \,.\, AO} = 1 + \frac{OL}{AB}.$$

$$\therefore \frac{1}{OM} = \frac{1}{OL} + \frac{1}{AB}.$$

678. Let C be the centre of the circle, BN the polar of A, N being on CA. The circle on AB passes through N, as $\angle ANB = 90°$; and the (tangent)^2 from C to this ⊙ $= CN \,.\, CA =$ (radius of C⊙)^2, ∴ the two ⊙s are orthogonal, ∴ the tangent from O to the C ⊙ $=$ OA.

679. Let KD, AC intersect in F; KA, BD in G.

$$X\{KCZD\} = \{KFZD\} = C\{KFZD\} = \{EALD\},$$

and $X\{KCYD\} = \{KAYG\} = B\{KAYG\} = \{EALD\}$.

∴ XZ, XY are collinear.

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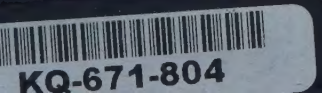
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